HOMOGENISATION FOR MAXWELL AND FRIENDS

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Abstract. We refine the understanding of continuous dependence on coefficients of solution operators under the nonlocal H-topology viz Schur topology in the setting of evolutionary equations in the sense of Picard. We show that certain components of the solution operators converge strongly. The weak convergence behaviour known from homogenisation problems for ordinary differential equations is recovered on the other solution operator components. The results are underpinned by a rich class of examples that, in turn, are also treated numerically, suggesting a certain sharpness of the theoretical findings. Analytic treatment of an example that proves this sharpness is provided too. Even though all the considered examples contain local coefficients, the main theorems and structural insights are of operator-theoretic nature and, thus, also applicable to nonlocal coefficients. The main advantage of the problem class considered is that they contain mixtures of type, potentially highly oscillating between different types of PDEs; a prototype can be found in Maxwell's equations highly oscillating between the classical equations and corresponding eddy current approximations.

1. Introduction

The theory of homogenisation addresses the effective behaviour of solutions of certain differential equations with highly oscillatory coefficients. The mathematical theory roots in the late 60s of the 20th century, and a standard account of the theory can be found in the seminal monographs [BLP78, ZKO94, Tar09]; with a more elementary introduction in [CD99]. With a focus on elliptic differential equations in variational form, a general viewpoint has led to the development of the notions of G- and H-convergence, see, e.g., [MT97, Tar09, CD99]. In these notions, the main object of study were sequences of certain matrix-valued multiplication operators. In order to understand homogenisation problems for evolutionary equations in the sense of Picard, a class of abstract operator equations providing a unified set-up for many time-dependent (partial) differential equations of mathematical physics, see [Pic09], a more operator-theoretic perspective needed to be advanced. We refer to [STW22, Chapters 13 and 14] and the references therein to get a picture of the first ten years of research concerning homogenisation theory for evolutionary equations. In the course of understanding this realm of questions in [Wau18], the notion of nonlocal H-convergence has been developed, which is a nonlocal generalisation of classical H-convergence, allowing for general operator coefficients. In turn, this led to the development of the Schur topology, see [NW22, Wau22, BSW23], with the main result in [BSW23] as culmination point, establishing continuous dependence results for evolutionary equations if the operator coefficients are endowed with a holomorphic variant of the Schur topology. Note that [BSW23] together with the compactness statement in [BEW24] (which in turn is a particular perspective given by the more general framework for Friedrichs systems in

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[BEW23]) provides a good understanding of homogenisation problems for evolutionary equations.

Even though the theoretical understanding is well-developed, a decent list of rather involved examples illustrating the theory is still missing. Thus, the first aim of the present article is to fill this gap. Moreover, the fundamental difference of assumptions in [BSW23] compared to the G-compactness statement in [BEW24] is a certain compactness condition, which is needed in the former and rather irrelevant for the latter. Hence, a second insight gathered here is that the compactness condition actually improves weak convergence to strong convergence on parts of the solution operator. Analytic treatment of one example will yield a decomposition into two infinite-dimensional subspaces such that one part converges strongly and the other is known to only converge weakly. Thirdly, we shall underpin our theoretical findings by treating all the examples also numerically. These numerical experiments additionally highlight our theoretical findings. The question, whether the part with strong convergence that we obtain is maximal, is an avenue open for future research.

We emphasise that the list of examples range from ordinary differential equations to higher dimensional partial differential equations, where the latter class also contains mixed type equations with highly oscillatory coefficients. In particular, an example for Maxwell's equations is treated, which, after the homogenisation process, triggers a memory effect in the limit equation, which is classical and can also be found in [Wel01, Wau16]. The list of examples stresses the versatility of the concept of evolutionary equations and the applicability of the main convergence result in [BSW23]. The Schur topology provides the precise setting enabling us to compute the effective evolutionary equations in a systematic manner. The numerics are based on [FTW19] developed precisely for mixed type problems written in the form of evolutionary equations.

We quickly summarise the organisation of the paper. In Section 2, we recall the general setting of evolutionary equations together with its corner stone Picard's Theorem 2.2. Section 3 serves to present a round up of results of homogenisation theory needed here. In particular, we recall the notion of nonlocal H-convergence and provide the main convergence result of [BSW23] together with the refinement concerning the quality of convergence. In Section 4 we state and prove some additional results concerning homogenisation theory that either have been overlooked so far or have only been announced in the literature. The list of examples can be found in Section 5; the corresponding numerical study is provided in Section 6. Section 7 contains a small conclusion. Some supplementary material can be found in Appendix A.

The Hilbert spaces that we will consider are anti-linear in the first and linear in the second argument. For a Hilbert space $\mathcal{H}$, a bounded linear operator $A \in \mathcal{L}_{\mathrm{b}}(\mathcal{H})$, and $c > 0$, we will write $\operatorname{Re} A \geq c$ instead of

$$\forall h \in \mathcal{H} : \operatorname{Re}\langle h, Ah\rangle_{\mathcal{H}} \geq c\langle h, h\rangle_{\mathcal{H}} = c\|h\|_{\mathcal{H}}^2,$$

and $\operatorname{Re} A > 0$ if we deem the exact knowledge of $c > 0$ unimportant. If we write $A\colon \operatorname{dom}(A) \subseteq \mathcal{H}_1 \to \mathcal{H}_2$ for Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, then A stands for a possibly unbounded operator with domain $\operatorname{dom}(A)$ and the adjoint A^* is considered with respect to $\mathcal{H}_1$ and $\mathcal{H}_2$.

2. The Setting: Evolutionary Equations

In this section, we recall the basic setting of evolutionary equations in the sense of Picard, [Pic09]. All results presented in the current section can be found with complete proofs in [STW22]. Let $\mathcal{H}$ be a Hilbert space, $\nu > 0$ and we set

$$\mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H}) \coloneqq \left\{ f \in \mathrm{L}_{1,\mathrm{loc}}(\mathbb{R};\mathcal{H}) \,\middle|\, \int_{\mathbb{R}} \|f(t)\|_{\mathcal{H}}^2 \mathrm{e}^{-2\nu t}\,\mathrm{d}t < \infty \right\}.$$

This is a Hilbert space endowed with the obvious scalar product. The Sobolev space of weakly differentiable functions $f \in \mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H})$ with distributional derivitative $f' \in \mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H})$ is denoted by $\mathrm{H}^1_\nu(\mathbb{R};\mathcal{H})$. Next, we define

$$\partial_{t,\nu} \colon \begin{cases} \mathrm{H}^1_\nu(\mathbb{R};\mathcal{H}) \subseteq \mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H}) & \to \quad \mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H}) \\ \qquad\qquad f & \mapsto \quad f'. \end{cases}$$

The Fourier–Laplace transformation $\mathcal{L}_\nu \in \mathcal{L}_\mathrm{b}(\mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H}), \mathrm{L}_2(\mathbb{R};\mathcal{H}))$ is the unitary extension of the mapping satisfying

$$(\mathcal{L}_\nu f)(\xi) \coloneqq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\xi t - \nu t} f(t)\,\mathrm{d}t \quad (\xi \in \mathbb{R}, f \in \mathrm{C_c}(\mathbb{R};\mathcal{H})).$$

This unitary transformation provides the spectral representation for $\partial_{t,\nu}$. Indeed, defining the multiplication-by-argument operator

$$\mathrm{m} \colon \begin{cases} \big\{ f \in \mathrm{L}_2(\mathbb{R};H) \,\big|\, (\xi \mapsto \xi f(\xi)) \in \mathrm{L}_2(\mathbb{R};\mathcal{H}) \big\} \subseteq \mathrm{L}_2(\mathbb{R};\mathcal{H}) & \to \quad \mathrm{L}_2(\mathbb{R};\mathcal{H}), \\ \qquad\qquad f & \mapsto \quad (\xi \mapsto \xi f(\xi)), \end{cases}$$

we obtain the equality

$$\partial_{t,\nu} = \mathcal{L}_\nu^* (\mathrm{im} + \nu) \mathcal{L}_\nu.$$

This equality serves as a means to define holomorphic functions of $\partial_{t,\nu}$. For this we introduce the following notion.

Definition 2.1. Let $M \colon \mathrm{dom}(M) \subseteq \mathbb{C} \to \mathcal{L}_\mathrm{b}(\mathcal{H})$. We call M a *material law*, if

(i) $\mathrm{dom}(M)$ is open, M is holomorphic and

(ii) there exists $\nu \in \mathbb{R}$ such that $\mathbb{C}_{\mathrm{Re}>\nu} \coloneqq \{z \in \mathbb{C} \,|\, \mathrm{Re}\, z > \nu\} \subseteq \mathrm{dom}(M)$ and

$$\|M\|_{\infty,\nu} \coloneqq \sup_{z \in \mathbb{C}_{\mathrm{Re}>\nu}} \|M(z)\| < \infty.$$

We set $s_b(M) \coloneqq \inf\{\nu \in \mathbb{R} \,|\, \text{(ii) holds}\}$ and call it the *abscissa of boundedness of* M. The set of all material laws with abscissa of boundedness lower than some $\mu \in \mathbb{R}$ is denoted by $\mathcal{M}(\mathcal{H}, \mu)$.

For a material law $M \in \mathcal{M}(\mathcal{H}, \nu)$, we furthermore define the corresponding *material law operator* $M(\partial_{t,\nu}) \in \mathcal{L}_\mathrm{b}(\mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H}))$ by

$$M(\partial_{t,\nu}) f \coloneqq \mathcal{L}_\nu^* M(\mathrm{im} + \nu) \mathcal{L}_\nu f \quad (f \in \mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H})),$$

where

$$(M(\mathrm{im} + \nu)\phi)(t) \coloneqq M(\mathrm{i}t + \nu)\phi(t) \quad (\phi \in \mathrm{L}_2(\mathbb{R};\mathcal{H}), \text{a.e. } t \in \mathbb{R}).$$ ♦

Next, we present the fundamental theorem for evolutionary equations, Picard's well-posedness theorem. For this, we do not use a different notation for a skew-selfadjoint operator acting on $\mathcal{H}$ and its (canonical) skew-selfadjoint extension to $\mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H})$. It will always be clear from the context, which operator is considered.

Theorem 2.2 (Picard's Theorem, [STW22, Theorem 6.2.1])**.** *Let $A \colon \mathrm{dom}(A) \subseteq \mathcal{H} \to \mathcal{H}$ be skew-selfadjoint, $\nu \in \mathbb{R}$, $M \in \mathcal{M}(\mathcal{H}, \nu)$. Assume there exists $c > 0$ such that*

$$\mathrm{Re}\, z M(z) \geq c \quad \textit{for all } z \in \mathbb{C}_{\mathrm{Re}\geq\nu}.$$

Then, the operator

$$\mathcal{B}_\nu \colon \begin{cases} \mathrm{H}^1_\nu(\mathbb{R};\mathcal{H}) \cap \mathrm{L}_{2,\nu}(\mathbb{R};\mathrm{dom}(A)) \subseteq \mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H}) & \to \quad \mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H}), \\ \qquad\qquad U & \mapsto \quad [\partial_{t,\nu} M(\partial_{t,\nu}) + A] U, \end{cases}$$

is closable. The closure is continuously invertible, $\mathcal{S}_\nu \coloneqq \overline{\mathcal{B}_\nu}^{-1}$. Morever, $\|\mathcal{S}_\nu\| \leq 1/c$.

Remark 2.3. (i) There are additional consequences of the assumptions in Picard's Theorem that we will not need but are worth mentioning. For instance, $\mathcal{S}_\nu$ is independent of the particular choice of ν. If $\mu \geq \nu$ and $f \in \mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H}) \cap \mathrm{L}_{2,\mu}(\mathbb{R};\mathcal{H})$ then $\mathcal{S}_\nu f = \mathcal{S}_\mu f$. Also, the solution operator $\mathcal{S}_\nu$ is *causal*; that is, for all $f, g \in \mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H})$ and $a \in \mathbb{R}$, $f = g$ on $(-\infty, a]$ implies $\mathcal{S}_\nu f = \mathcal{S}_\nu g$ $(-\infty, a]$. Furthermore, we have the following regularity statement: if $f \in \mathrm{H}^1_\nu(\mathbb{R};\mathcal{H})$, then $\mathcal{S}_\nu f \in \mathrm{H}^1_\nu(\mathbb{R}; H) \cap \mathrm{L}_{2,\nu}(\mathbb{R}; \mathrm{dom}(A))$.

(ii) The positive-definiteness condition on the material law implies that, for all $z \in \mathbb{C}_{\mathrm{Re}\geq\nu}$, the operator $(zM(z) + A)$ is continuously invertible on $\mathcal{H}$ with the norm of the inverse bounded by $1/c$. In fact, it is only this uniformly bounded invertibility that is used to obtain the numerous conclusions. Thus, instead of the positive definiteness condition, it suffices to assume that there exists $c > 0$ such that

$$\sup_{z\in\mathbb{C}_{\mathrm{Re}\geq\nu}} \|(zM(z) + A)^{-1}\| \leq 1/c.$$

Then, the same conclusions as in Picard's Theorem (also the ones mentioned in item (i)) hold.

(iii) As mentioned in item (ii), the proof of Picard's Theorem hinges upon having $(zM(z) + A)^{-1} \in \mathcal{L}_{\mathrm{b}}(\mathcal{H})$ for $z \in \mathbb{C}_{\mathrm{Re}\geq\nu}$. In fact, one shows that this defines a material law with corresponding operator $\mathcal{S}_\nu$. ♦

For $d \in \mathbb{N}$ and an open $\Omega \subseteq \mathbb{R}^d$, we define $\mathrm{grad}\restriction_{\mathrm{C}_c^\infty} : \mathrm{C}_c^\infty(\Omega) \subseteq \mathrm{L}_2(\Omega) \to \mathrm{L}_2(\Omega)^d$ and $\mathrm{div}\restriction_{\mathrm{C}_c^\infty} : \mathrm{C}_c^\infty(\Omega)^d \subseteq \mathrm{L}_2(\Omega)^d \to \mathrm{L}_2(\Omega)$ the usual way. We also get $\mathrm{curl}\restriction_{\mathrm{C}_c^\infty} : \mathrm{C}_c^\infty(\Omega)^3 \subseteq \mathrm{L}_2(\Omega)^3 \to \mathrm{L}_2(\Omega)^3$. As usual, we can weakly extend these unbounded operators to their maximal domains, i.e., $\mathrm{grad} \coloneqq -(\mathrm{div}\restriction_{\mathrm{C}_c^\infty})^*$, $\mathrm{div} \coloneqq -(\mathrm{grad}\restriction_{\mathrm{C}_c^\infty})^*$ and $\mathrm{curl} \coloneqq (\mathrm{curl}\restriction_{\mathrm{C}_c^\infty})^*$. Adjoining again, we get the operators with zero boundary conditions $\mathring{\mathrm{grad}} \coloneqq -\mathrm{div}^* = \overline{\mathrm{grad}\restriction_{\mathrm{C}_c^\infty}}$, $\mathring{\mathrm{div}} \coloneqq -\mathrm{grad}^* = \overline{\mathrm{div}\restriction_{\mathrm{C}_c^\infty}}$ and $\mathring{\mathrm{curl}} \coloneqq \mathrm{curl}^* = \overline{\mathrm{curl}\restriction_{\mathrm{C}_c^\infty}}$. Since all these extended operators are closed, their domains are Hilbert spaces if endowed with the respective graph product. Most prominently, we get the Sobolev spaces $\mathrm{H}^1(\Omega)$ and $\mathring{\mathrm{H}}^1(\Omega)$ stemming from the domains of grad and $\mathring{\mathrm{grad}}$ respectively, and similarly, $\mathrm{H}(\mathrm{div}, \Omega)$, $\mathring{\mathrm{H}}(\mathrm{div}, \Omega)$, $\mathrm{H}(\mathrm{curl}, \Omega)$ and $\mathring{\mathrm{H}}(\mathrm{curl}, \Omega)$.

Example 2.4 (Maxwell's Equations)**.** For $\Omega \subseteq \mathbb{R}^3$ open, $\mathcal{H} \coloneqq \mathrm{L}_2(\Omega)^3 \times \mathrm{L}_2(\Omega)^3$, $c > 0$ and $\nu_0 > 0$, let the bounded and measurable dielectric permittivity, magnetic permeability and electric conductivity $\epsilon, \mu, \sigma : \Omega \to \mathbb{R}^{3\times 3}$ satisfy

$$\forall x \in \Omega : \mu(x) = \mu(x)^* \geq c \text{ and } \nu\epsilon(x) + \mathrm{Re}\,\sigma(x) = \nu\epsilon(x)^* + \mathrm{Re}\,\sigma(x) \geq c$$

for all $\nu \geq \nu_0$. Maxwell's equations for the electric and magnetic field, E and H, with current $j_0 \in \mathrm{dom}(\partial_{t,\nu})$ and perfect conductor boundary conditions read

$$\left[\partial_{t,\nu}\begin{pmatrix}\epsilon & 0\\ 0 & \mu\end{pmatrix} + \begin{pmatrix}\sigma & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & -\mathrm{curl}\\ \mathring{\mathrm{curl}} & 0\end{pmatrix}\right]\begin{pmatrix}E\\ H\end{pmatrix} = \begin{pmatrix}j_0\\ 0\end{pmatrix}.$$

Note that all conditions of Theorem 2.2 are met. We emphasise that due to positive parameter $\nu > 0$, we ask for an implicit homogeneous initial condition at $-\infty$. Classical initial value problems can also be treated, see [STW22, Chapter 9]. ♦

3. Homogenisation Theory and Theorems

In this section, we shall add some more structural and less computational points to this subject matter. In particular, we will recall the classical (local) homogenisation theory from [MT97] and [Tar09], we will introduce the notion of nonlocal H-convergence (and, thus, the Schur topology) from [Wau18] (generalised in [Wau22]), and we will refine the (nonlocal) homogenisation result for evolutionary equations from [BSW23].

In Section 2, we have introduced div weakly on its maximal domain in $L_2(\Omega)^d$, where $\Omega \subseteq \mathbb{R}^d$ open and $d \in \mathbb{N}$. We will extend this operator canonically to $\operatorname{div}_{-1}\colon L_2(\Omega)^d \to H^{-1}(\Omega)$, where $H^{-1}(\Omega)$ is the dual space of $\mathring{H}^1(\Omega)$, via $\operatorname{div}_{-1}(\varphi)(u) := -\langle \varphi, \mathring{\operatorname{grad}} u\rangle_{L_2(\Omega)^d}$ for all $\varphi \in L_2(\Omega)^d$ and $u \in \mathring{H}^1(\Omega)$. Clearly, div_{-1} is bounded and anti-linear.

3.1. **Local H-Convergence.** Classical PDE-homogenisation theory is tailored to the sequence of unique solutions $(u_n)_n$ in $\mathring{H}^1(\Omega), n \in \mathbb{N}$, of

$$-\operatorname{div}_{-1} a_n \mathring{\operatorname{grad}} u_n = f \tag{1}$$

where, for some $d \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^d$ is bounded and open, $f \in H^{-1}(\Omega)$, and $(a_n)_{n\in\mathbb{N}}$ is a given sequence of linear bounded operators on $L_2(\Omega)^d$ with $\operatorname{Re} a_n \geq c$ for all $n \in \mathbb{N}$ and some $c > 0$. In case that $(u_n)_{n\in\mathbb{N}}$ at least weakly converges to some $u \in \mathring{H}^1$, one would like to obtain an $a \in \mathcal{L}_\mathrm{b}(L_2(\Omega)^d)$ with $\operatorname{Re} a \geq \tilde{c}$ for some $\tilde{c} > 0$ such that u solves

$$-\operatorname{div}_{-1} a \mathring{\operatorname{grad}} u = f, \tag{2}$$

and one would like this a to be independent of f. At this point, we have to note two things. Firstly, this convergence is apparently more a condition on the coefficient sequence $(a_n)_{n\in\mathbb{N}}$ than on the solutions (which clearly depend on f). Secondly, it turns out that neither existence nor uniqueness of a limit in terms of $(a_n)_{n\in\mathbb{N}}$ is guaranteed in this general setting.[1] In the special case of multiplication (and thus local) operators $a_n, a \in L_\infty(\Omega)^{d\times d}, n \in \mathbb{N}$, [MT97] and [Tar09] now provide the following theory.

Definition 3.1 ([Tar09, Definition 6.4])**.** Consider the set

$$M(\alpha,\beta,\Omega) := \left\{a \in L_\infty(\Omega)^{d\times d} \,\middle|\, \operatorname{Re} a(x) \geq \alpha, \operatorname{Re} a(x)^{-1} \geq \beta \text{ for } x \in \Omega \text{ a.e.}\right\} \tag{3}$$

for some $d \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^d$ bounded and open, and for some $0 < \alpha < \beta$.

We call a sequence $(a_n)_{n\in\mathbb{N}}$ from $M(\alpha,\beta,\Omega)$ *locally* H*-convergent* to an $a \in M(\alpha,\beta,\Omega)$ if and only if for all $f \in H^{-1}(\Omega)$ the sequence $(u_n)_{n\in\mathbb{N}}$ of $\mathring{H}^1$-solutions of (1) weakly converges to the solution $u \in \mathring{H}^1(\Omega)$ of (2) in $\mathring{H}^1(\Omega)$, and $a_n \mathring{\operatorname{grad}} u_n$ weakly converges to $a \mathring{\operatorname{grad}} u$ in $L_2(\Omega)^d$, i.e.,

$$u_n \rightharpoonup u \text{ in } \mathring{H}^1(\Omega) \quad \text{and} \quad a_n \mathring{\operatorname{grad}} u_n \rightharpoonup a \mathring{\operatorname{grad}} u \text{ in } L_2(\Omega). \qquad \blacklozenge$$

Theorem 3.2 ([Tar09, p. 82 (metric topology) and Theorem 6.5 (compact)])**.** *For $d \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^d$ bounded and open, and $0 < \alpha < \beta$, there exists a topology τ on $M(\alpha,\beta,\Omega)$ that induces local* H*-convergence and renders the space $(M(\alpha,\beta,\Omega),\tau)$ Hausdorff, compact, and even metrisable.*

In certain cases, we can compute the homogenisation limit explicitly.

Theorem 3.3 ([CD99, Theorem 5.12])**.** *Let $\Omega \subseteq \mathbb{R}^d$ be open and bounded for some $d \in \mathbb{N}$ and $a = (a_{ij})_{i,j=1}^d \in M(\alpha,\beta,\Omega)$ given by bounded and measurable ℓ_1-periodic functions*

$$\hat{a}_{ij}\colon \mathbb{R} \to \mathbb{R},$$

[1] Actually, existence follows by Theorem 3.2 if we restrict the coefficient sequence to the set (3) of multiplication operators. This leads to the older concept of G-convergence (H-convergence implies G-convergence) introduced by Sergio Spagnolo in the 1960s. Alternatively, Theorem 3.6 below can be viewed as replacing the restriction to (3) by requiring mere boundedness of the sequence $(a_n)_{n\in\mathbb{N}}$, i.e., a nonlocal generalisation. Uniqueness is only achievable in special cases since the solution u of (2) disregards constant real skew-selfadjoint additive perturbations of a (cf. Lemma A.1). For an in-depth discussion see, e.g., [Tar09, Chapter 6].

where $a_{ij}\colon \Omega \to \mathbb{R}$ *and* $a_{ij}(x) = \hat{a}_{ij}(x_1)$ *for* $x = (x_1, \dots, x_d) \in \Omega$*. For* $n \in \mathbb{N}$*, we define* $a_n(x) \coloneqq a(nx)$ *and the corresponding sequence* $(a_n)_{n\in\mathbb{N}}$*. Then for every* $f \in \mathrm{H}^{-1}(\Omega)$*, the solution sequence* $(u_n)_{n\in\mathbb{N}}$ *to the problems*

$$-\operatorname{div}_{-1} a_n \overset{\circ}{\operatorname{grad}} u_n = f \tag{4}$$

converges weakly to u_{hom} *in* $\overset{\circ}{\mathrm{H}}{}^1(\Omega)$*, where* u_{hom} *is the solution of*

$$-\operatorname{div}_{-1} a_{\mathrm{hom}} \overset{\circ}{\operatorname{grad}} u_{\mathrm{hom}} = f,$$

and the matrix-valued function $a_{\mathrm{hom}} \in M(\alpha, \beta, \Omega)$ *is given by*

$$\begin{aligned}
(a_{\mathrm{hom}})_{11} &= \frac{1}{\mathrm{m}(\frac{1}{a_{11}})}, \\
(a_{\mathrm{hom}})_{1j} &= (a_{\mathrm{hom}})_{11} \mathrm{m}\Big(\frac{a_{1j}}{a_{11}}\Big), && 2 \le j \le d \\
(a_{\mathrm{hom}})_{i1} &= (a_{\mathrm{hom}})_{11} \mathrm{m}\Big(\frac{a_{i1}}{a_{11}}\Big), && 2 \le i \le d \\
(a_{\mathrm{hom}})_{ij} &= (a_{\mathrm{hom}})_{11} \mathrm{m}\Big(\frac{a_{i1}}{a_{11}}\Big)\mathrm{m}\Big(\frac{a_{1j}}{a_{11}}\Big) + \mathrm{m}\Big(a_{ij} - \frac{a_{i1}a_{1j}}{a_{11}}\Big), && 2 \le i, j \le d,
\end{aligned}$$

where $\mathrm{m}(b)$ *is the integral mean over* $b \in \mathrm{L}_1(0, \ell_1)$*, i.e.,* $\mathrm{m}(b) \coloneqq \frac{1}{\ell_1}\int_0^{\ell_1} b(z)\,\mathrm{d}z$*. Additionally, the sequence* $(a_n \overset{\circ}{\operatorname{grad}} u_n)_{n\in\mathbb{N}}$ *converges weakly to* $a_{\mathrm{hom}} \overset{\circ}{\operatorname{grad}} u_{\mathrm{hom}}$ *in* $\mathrm{L}_2(\Omega)^d$*, i.e.,*

$$a_n \overset{\circ}{\operatorname{grad}} u_n \rightharpoonup a_{\mathrm{hom}} \overset{\circ}{\operatorname{grad}} u_{\mathrm{hom}} \quad \text{in} \quad \mathrm{L}_2(\Omega)^d. \tag{5}$$

All in all, this means $(a_n)_{n\in\mathbb{N}}$ *locally* H*-converges to* a_{hom}*.*

Remark 3.4. Note the following:

- The last claim (5) is not part of the original statement, but it is shown in the course of proof, see [CD99, Proof of Thm. 5.10].
- Furthermore, [CD99] defines $M(\alpha, \beta, \Omega)$ without the real part operator, since only real vector spaces are treated. However, if a is an $\mathbb{R}^{d\times d}$-valued function, then both approaches to $M(\alpha, \beta, \Omega)$ conincide (theirs and ours), see Lemma A.2. ♦

3.2. **Nonlocal H-Convergence.** In [Wau18], the desire to treat general (possibly nonlocal) coefficients instead of only multiplication operators, and to treat a whole class of (possibly time-dependent) systems at once led to the following operator-theoretic approach to homogenisation.

For the rest of this section, let $\mathcal{H}$ be a separable Hilbert space that can be decomposed orthogonally into two closed subspaces $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Hence, any $a \in \mathcal{L}_{\mathrm{b}}(\mathcal{H})$ can equivalently be written as $\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$ where $a_{ij} \in \mathcal{L}_{\mathrm{b}}(\mathcal{H}_j, \mathcal{H}_i)$.

Definition 3.5 (cf. [Wau18, Theorem 4.1])**.** Consider

$$\mathcal{M}(\mathcal{H}_0, \mathcal{H}_1) \coloneqq \{a \in \mathcal{L}_{\mathrm{b}}(\mathcal{H}) \mid a_{00}^{-1} \in \mathcal{L}_{\mathrm{b}}(\mathcal{H}_0) \text{ and } a^{-1} \in \mathcal{L}_{\mathrm{b}}(\mathcal{H})\}.$$

A sequence $(a_n)_{n\in\mathbb{N}}$ from $\mathcal{M}(\mathcal{H}_0, \mathcal{H}_1)$ is said to *nonlocally* H*-converge* to an $a \in \mathcal{M}(\mathcal{H}_0, \mathcal{H}_1)$ if and only if $a_{n,00}^{-1}$, $a_{n,10}a_{n,00}^{-1}$, $a_{n,00}^{-1}a_{n,01}$, and $a_{n,11} - a_{n,10}a_{n,00}^{-1}a_{n,01}$ converge pointwise weakly (i.e., with respect to the weak operator topology) to a_{00}^{-1}, $a_{10}a_{00}^{-1}$, $a_{00}^{-1}a_{01}$, and $a_{11} - a_{10}a_{00}^{-1}a_{01}$ respectively. ♦

Theorem 3.6 (cf. [Wau18, Chapter 5])**.** *There exists a topology* $\tau(\mathcal{H}_0, \mathcal{H}_1)$ *on* $\mathcal{M}(\mathcal{H}_0, \mathcal{H}_1)$*, called* nonlocal H-topology *or* Schur topology*, that induces nonlocal* H*-convergence and* *renders the space* $(\mathcal{M}(\mathcal{H}_0,\mathcal{H}_1),\tau(\mathcal{H}_0,\mathcal{H}_1))$ *Hausdorff. Furthermore, regard*

$$\mathcal{M}(\gamma,\mathcal{H}_0,\mathcal{H}_1) := \Big\{ a\in\mathcal{M}(\mathcal{H}_0,\mathcal{H}_1) \,\Big|\, \operatorname{Re} a_{00} \geq \gamma_{00}, \operatorname{Re} a_{00}^{-1} \geq \frac{1}{\gamma_{11}}, \|a_{10}a_{00}^{-1}\| \leq \gamma_{10}, \|a_{00}^{-1}a_{01}\| \leq \gamma_{01}, \operatorname{Re}(a_{11}-a_{10}a_{00}^{-1}a_{01})^{-1} \geq \frac{1}{\gamma_{11}}, \operatorname{Re}(a_{11}-a_{10}a_{00}^{-1}a_{01}) \geq \gamma_{00} \Big\},$$

where $\gamma = \begin{pmatrix}\gamma_{00} & \gamma_{01}\\ \gamma_{10} & \gamma_{11}\end{pmatrix} \in (0,\infty)^{2\times 2}$*. Then,* $(\mathcal{M}(\gamma,\mathcal{H}_0,\mathcal{H}_1),\tau(\mathcal{H}_0,\mathcal{H}_1))$ *is compact and metrisable.*

In the following, if the decomposition of $\mathcal{H}$ into $\mathcal{H}_0$ and $\mathcal{H}_1$ is clear from the context, we will simply write $\mathcal{M}(\gamma)$ instead of $\mathcal{M}(\gamma,\mathcal{H}_0,\mathcal{H}_1)$. As one would expect, nonlocal H-convergence reasonably generalises local H-convergence. For the matter of simplicity, we will only treat topologically trivial domains (in 3D: simply connected with connected complement). For the general case, see [Wau22].

Remark 3.7 (Helmholtz decomposition, [Wau18, Examples 2.3 and 2.4]). For a bounded simply connected weak Lipschitz domain $\Omega\subseteq\mathbb{R}^3$ with connected complement and continuous boundary, we have the following orthogonal decompositions into closed subspaces

$$\mathrm{L}_2(\Omega)^3 = \operatorname{ran}(\overset{\circ}{\operatorname{grad}}) \oplus \operatorname{ran}(\operatorname{curl}) = \operatorname{ran}(\operatorname{grad}) \oplus \operatorname{ran}(\overset{\circ}{\operatorname{curl}}),$$

where $\Omega^{\complement}$ being connected implies the first decomposition, and Ω being simply connected the second one. In other words, we have $\operatorname{ran}(\operatorname{grad}) = \ker(\operatorname{curl})$, $\operatorname{ran}(\overset{\circ}{\operatorname{grad}}) = \ker(\overset{\circ}{\operatorname{curl}})$, $\ker(\operatorname{div}) = \operatorname{ran}(\operatorname{curl})$ and $\ker(\overset{\circ}{\operatorname{div}}) = \operatorname{ran}(\overset{\circ}{\operatorname{curl}})$. Furthermore, the Picard-Weber-Weck selection theorem holds ([Pic84]): $\operatorname{dom}(\operatorname{div})\cap\operatorname{dom}(\overset{\circ}{\operatorname{curl}})$ and $\operatorname{dom}(\overset{\circ}{\operatorname{div}})\cap\operatorname{dom}(\operatorname{curl})$, both endowed with the inner product $\langle\cdot,\cdot\rangle_{\mathrm{L}_2(\Omega)^3} + \langle\operatorname{div}\cdot,\operatorname{div}\cdot\rangle_{\mathrm{L}_2(\Omega)} + \langle\operatorname{curl}\cdot,\operatorname{curl}\cdot\rangle_{\mathrm{L}_2(\Omega)^3}$, are each compactly embedded into $\mathrm{L}_2(\Omega)^3$. ♦

Theorem 3.8 ([Wau18, Theorem 5.11])**.** *Let $\Omega\subseteq\mathbb{R}^3$ be a bounded weak Lipschitz domain with connected complement. For $0<\alpha<\beta$, a sequence $(a_n)_{n\in\mathbb{N}}$ from $M(\alpha,\beta,\Omega)$ locally* H*-converges to some $a\in M(\alpha,\beta,\Omega)$ if and only if it* $\tau(\operatorname{ran}(\overset{\circ}{\operatorname{grad}}),\operatorname{ran}(\operatorname{curl}))$*-converges to* a.

3.3. **Homogenisation Theory for Evolutionary Equations.** In the remaining section, assume that $\mathcal{H}$ is a separable Hilbert space and $A\colon \operatorname{dom}(A)\subseteq\mathcal{H}\to\mathcal{H}$ skew-selfadjoint. Additionally, let the Hilbert space $\operatorname{dom}(A)\cap(\ker A)^{\perp}$ (with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}} + \langle A\cdot,A\cdot\rangle_{\mathcal{H}}$) be compactly embedded into $\mathcal{H}$. This implies $\ker(A)^{\perp} = \operatorname{ran} A$ (see, e.g., [tEGW19, Lemma 4.1] or the FA-Toolbox in [PZ23]), i.e., $\mathcal{H} = \mathcal{H}_0\oplus\mathcal{H}_1$ in the sense of Section 3.2 with the Hilbert spaces $\mathcal{H}_0 := \ker(A)$ and $\mathcal{H}_1 := \operatorname{ran}(A)$. Looking at the proof of [BSW23, Lemma 6.4], we see that actually the following stronger statement was shown.

Lemma 3.9. *Let $\gamma\in(0,\infty)^{2\times2}$ and consider a sequence $(T_n)_{n\in\mathbb{N}}$ from $\mathcal{M}(\gamma)$ that* $\tau(\ker(A),\operatorname{ran}(A))$*-converges to some $T\in\mathcal{M}(\gamma)$. Then, $(T_n+A)^{-1},(T+A)^{-1}\in\mathcal{L}_{\mathrm{b}}(\mathcal{H})$, for all $n\in\mathbb{N}$, and for $\varphi\in\mathcal{H}$,*

- *the* $\ker(A)$*-component of $(T_n+A)^{-1}\varphi$ weakly converges to $(T+A)^{-1}\varphi$,*
- *the* $\operatorname{ran}(A)$*-component of $(T_n+A)^{-1}\varphi$ converges to $(T+A)^{-1}\varphi$.*

Apparantly, by (canonically) extending the orthogonal projections from $\mathcal{H}$ to $\mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H})$, the decomposition of $\mathcal{H}$ can be lifted to

$$\mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H}) = \mathrm{L}_{2,\nu}(\mathbb{R};\ker(A)) \oplus \mathrm{L}_{2,\nu}(\mathbb{R};\operatorname{ran}(A)),$$

where $\nu \in \mathbb{R}$ is arbitrary.

Employing this and Lemma 3.9, we immediately obtain the following refinement of the homogenisation theorem [BSW23, Theorem 6.5]:

Theorem 3.10. *Consider $\nu_0 > 0$ and material laws M_n with $\mathbb{C}_{\mathrm{Re}>\nu_0} \subseteq \operatorname{dom}(M_n)$ for $n \in \mathbb{N}$. Let there be some $c, d > 0$ such that for all $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$ we have $\operatorname{Re} zM_n(z) \geq c$ and $\|M_n(z)\| \leq d$. Then, $M_n \in \mathcal{M}(\mathcal{H},\nu)$ for all $n \in \mathbb{N}$ and $\nu > \nu_0$. Assume there exists $M\colon \mathbb{C}_{\mathrm{Re}>\nu_0} \to \mathcal{M}(\ker(A),\operatorname{ran}(A))$ with $\|M(z)\| \leq d$ on $\mathbb{C}_{\mathrm{Re}>\nu_0}$. If $(M_n)_{n\in\mathbb{N}}$ pointwise $\tau(\ker(A),\operatorname{ran}(A))$-converges to M on $\mathbb{C}_{\mathrm{Re}>\nu_0}$, then $M \in \mathcal{M}(\mathcal{H},\nu)$ for all $\nu > \nu_0$ as well as $\operatorname{Re} zM(z) \geq c$ on $\mathbb{C}_{\mathrm{Re}>\nu_0}$. Most importantly, for $f \in \mathrm{L}_{2,\nu}(\mathbb{R},\mathcal{H})$, $\nu > \nu_0$, $n \in \mathbb{N}$, $\mathcal{S}_{n,\nu} \coloneqq \overline{[\partial_{t,\nu} M_n(\partial_{t,\nu}) + A]}^{-1}$ and $\mathcal{S}_\nu \coloneqq \overline{[\partial_{t,\nu} M(\partial_{t,\nu}) + A]}^{-1}$,*

- *the $\mathrm{L}_{2,\nu}(\mathbb{R};\ker(A))$-component of $\mathcal{S}_{n,\nu}f$ weakly converges to $\mathcal{S}_\nu f$,*
- *the $\mathrm{L}_{2,\nu}(\mathbb{R};\operatorname{ran}(A))$-component of $\mathcal{S}_{n,\nu}f$ converges to $\mathcal{S}_\nu f$.*

The previous theorem will be a crucial tool to justify the convergence of the examples in Section 5. In fact, by means of one analytic and several numerical examples we shall see that the convergence statements seem to be optimal.

Remark 3.11. Consider the setting of Theorem 3.10. Whether $\mathrm{L}_{2,\nu}(\mathbb{R};\operatorname{ran}(A))$ is a maximal (disregarding the trivial finite-dimensional extensions) subspace, on which $(\mathcal{S}_{n,\nu}f)_{n\in\mathbb{N}}$ converges to $\mathcal{S}_\nu f$ in norm for each $f \in \mathrm{L}_{2,\nu}(\mathbb{R},\mathcal{H})$, remains an open question. ♦

4. Additional Properties of (Non-)Local H-Convergence

In this section, we will prove a few, mostly straightforward, properties of local and nonlocal H-convergence that we will need in Section 5, and that only implicitly appear in the literature.

4.1. Nonlocal H-Convergence of the Inverse Sequence.

A frequent scenario one is confronted with (and that also appears in Section 5) is knowing how to compute the $\tau(\mathcal{H}_1,\mathcal{H}_0)$-limit but rather needing the $\tau(\mathcal{H}_0,\mathcal{H}_1)$-limit. The following theorem offers a solution: invert the sequence, compute the limit, and then invert back. This observation originated in [Wau22].

Theorem 4.1. *For a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathcal{M}(\mathcal{H}_0,\mathcal{H}_1)$, the sequence of inverses $((a_n)^{-1})_{n\in\mathbb{N}}$ lies in $\mathcal{M}(\mathcal{H}_1,\mathcal{H}_0)$. Furthermore, $\tau(\mathcal{H}_0,\mathcal{H}_1)$-convergence of $(a_n)_{n\in\mathbb{N}}$ to an $a \in \mathcal{M}(\mathcal{H}_0,\mathcal{H}_1)$ is equivalent to $\tau(\mathcal{H}_1,\mathcal{H}_0)$-convergence of $((a_n)^{-1})_{n\in\mathbb{N}}$ to a^{-1}.*

Proof. By [Wau18, Lemma 4.8], we obtain $(a_n)^{-1} \in \mathcal{M}(\mathcal{H}_1,\mathcal{H}_0)$ with the decomposition

$$(a_n)^{-1} = \begin{pmatrix} a_{n,00}^{-1} + a_{n,00}^{-1} a_{n,01} \tilde{a}_n a_{n,10} a_{n,00}^{-1} & -a_{n,00}^{-1} a_{n,01} \tilde{a}_n \\ -\tilde{a}_n a_{n,10} a_{n,00}^{-1} & \tilde{a}_n \end{pmatrix}$$

for $n \in \mathbb{N}$, where $\tilde{a}_n \coloneqq (a_{n,11} - a_{n,10} a_{n,00}^{-1} a_{n,01})^{-1}$. The definition of the respective Schur topologies now directly implies the statement. ❑

4.2. A Pasting Theorem.

In Section 5, we will also need a kind of a pasting property of local H-convergence. That means we want to conclude local H-convergence on a domain from local H-convergence on each of two disjoint subdomains that together span the domain in a certain topological and measure theoretical sense. This is basically an immediate corollary of [Tar09, Lemma 10.5].

Theorem 4.2. *Consider $d \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^d$ bounded and open, $0 < \alpha < \beta$, a sequence $(a_n)_{n\in\mathbb{N}}$ in $M(\alpha, \beta, \Omega)$, and some $a \in M(\alpha, \beta, \Omega)$. If for almost every $x \in \Omega$ there exists an open neighborhood $\omega_x \subseteq \Omega$ of x such that $(a_n \restriction_{\omega_x})_{n\in\mathbb{N}}$ converges to $a \restriction_{\omega_x}$ in $M(\alpha, \beta, \omega_x)$, then $(a_n)_{n\in\mathbb{N}}$ H-converges to a in $M(\alpha, \beta, \Omega)$.*

Proof. By Theorem 3.2, every subsequence of $(a_n)_{n\in\mathbb{N}}$ has another subsequence that converges to some $b \in M(\alpha, \beta, \Omega)$. From [Tar09, Lemma 10.5] and from the assumptions, we infer that a and b coincide. As the initial subsequence was chosen arbitrarily, this concludes the proof. ❑

4.3. **Independence of Boundary Conditions.** Another property that we will need in Section 5 is a variant of Theorem 3.8 with different boundary conditions that was already proposed in [Wau18, Remark 5.12], see also [BSW23, Theorem 2.4]. It hinges upon the independence of the homogeneous Dirichlet boundary conditions that were imposed in the definition of local H-convergence (cf. Section 3.1). To be precise, the following concrete realisation of [Tar09, Lemma 10.3] holds.

Lemma 4.3. *Consider $d \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^d$ bounded and open, and $0 < \alpha < \beta$. Then, for a sequence $(a_n)_{n\in\mathbb{N}}$ in $M(\alpha, \beta, \Omega)$ and $a \in M(\alpha, \beta, \Omega)$, the following statements are equivalent:*

(i) $(a_n)_{n\in\mathbb{N}}$ *locally* H-*converges to* a.

(ii) *For all $f \in \mathrm{H}^1_\perp(\Omega)'$, where $\mathrm{H}^1_\perp(\Omega) \coloneqq \{v \in \mathrm{H}^1(\Omega) \mid \int_\Omega v = 0\}$ (endowed with the $\mathrm{H}^1(\Omega)$-scalar product), the unique solutions $u_n \in \mathrm{H}^1_\perp(\Omega), n \in \mathbb{N}$, of*
$$\forall v \in \mathrm{H}^1_\perp(\Omega)\colon \langle a_n \operatorname{grad} u_n, \operatorname{grad} v\rangle_{\mathrm{L}_2(\Omega)^d} = f(v) \tag{6}$$
weakly $\mathrm{H}^1_\perp(\Omega)$-converge to the unique solution $u \in \mathrm{H}^1_\perp(\Omega)$ of
$$\forall v \in \mathrm{H}^1_\perp(\Omega)\colon \langle a \operatorname{grad} u, \operatorname{grad} v\rangle_{\mathrm{L}_2(\Omega)^d} = f(v), \tag{7}$$
and $a_n \operatorname{grad} u_n$ weakly $\mathrm{L}_2(\Omega)^d$-converges to $a \operatorname{grad} u$.

Proof. First, the same construction that eventually proves metrisability in Theorem 3.2 also shows that the convergence in (ii) is induced by a Hausdorff topology on $M(\alpha, \beta, \Omega)$.

Next, assume (i), and consider the solution sequence $(u_n)_{n\in\mathbb{N}}$ from (6) for some fixed f. Extract any subsequence, and use the same notation. By classical elliptic theory, this sequence is bounded in $\mathrm{H}^1(\Omega)$ by the norm of f times a constant that only depends on α and the shape of Ω. Thus, the sequence $(a_n \operatorname{grad} u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}_2(\Omega)^d$ by the same bound times $1/\beta$, and we obtain yet another subsequence, $u \in \mathrm{H}^1(\Omega)$, as well as $q \in \mathrm{L}_2(\Omega)^d$ such that $u_n \rightharpoonup u$ in $\mathrm{H}^1(\Omega)$, which also implies $u \in \mathrm{H}^1_\perp(\Omega)$, as well as $a_n \operatorname{grad} u_n \rightharpoonup q$ in $\mathrm{L}_2(\Omega)^d$. For $w \in \mathring{\mathrm{H}}^1(\Omega)$, we can calculate
$$\langle a_n \operatorname{grad} u_n, \operatorname{grad} w\rangle_{\mathrm{L}_2(\Omega)^d} = f(w - w_\Omega),$$
where $w_\Omega \coloneqq |\Omega|^{-1} \int_\Omega w \,\mathrm{d}x$ is the integral mean. Straightforward computations show $(w \mapsto f(w - w_\Omega)) \in \mathrm{H}^{-1}(\Omega)$. Therefore, we can apply [Tar09, Lemma 10.3] which yields $q = a \operatorname{grad} u$. It remains to verify (7), but this is an immediate consequence of $a_n \operatorname{grad} u_n \rightharpoonup a \operatorname{grad} u$ in $\mathrm{L}_2(\Omega)^d$.

So, the identity mapping is continuous from the compact local H-topology to the Hausdorff topology that induces the convergence in (ii). Hence, these two topologies on $M(\alpha, \beta, \Omega)$ coincide. ❑

With this, [Wau18, Remark 5.12] yields:

Theorem 4.4. *Let $\Omega \subseteq \mathbb{R}^3$ be a simply connected bounded weak Lipschitz domain. For $0 < \alpha < \beta$, a sequence $(a_n)_{n\in\mathbb{N}}$ from $M(\alpha, \beta, \Omega)$ locally* H-*converges to an $a \in M(\alpha, \beta, \Omega)$ if and only if it $\tau(\operatorname{ran}(\operatorname{grad}), \operatorname{ran}(\mathring{\operatorname{curl}}))$-converges to a.*

4.4. H**-convergence in 2D.** Finally, we will apply Theorem 3.8 in two dimensions in Section 5. Fortunately, this easily turns out to work exactly as expected and is in fact very similar to the 3D case.

Remark 4.5 (Helmholtz Decomposition in 2D). Let $\Omega \subseteq \mathbb{R}^2$ be open and simply connected with the segment property. Define $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then, it is known that[2]

$$\mathrm{L}_2(\Omega)^2 = \operatorname{ran}(\mathring{\operatorname{grad}}) \oplus \operatorname{ran}(J \operatorname{grad}) = \operatorname{ran}(\operatorname{grad}) \oplus \operatorname{ran}(J \mathring{\operatorname{grad}}). \tag{8}$$

Since this statement seems to be difficult to find (or follows from more involved statements involving Betti numbers and differential forms, see, e.g., [DS52, Pic79]), we provide a short argument here.

Clearly, we have (see also [Pau15, Appendix])

$$\mathrm{L}_2(\Omega)^2 = \operatorname{ran}(\mathring{\operatorname{grad}}) \oplus \operatorname{ran}(J \operatorname{grad}) \oplus \mathcal{H}_{\mathrm{D}} = \operatorname{ran}(\operatorname{grad}) \oplus \operatorname{ran}(J \mathring{\operatorname{grad}}) \oplus \mathcal{H}_{\mathrm{N}},$$

where

$$\mathcal{H}_{\mathrm{D}} := \ker(\operatorname{div}) \cap \ker(\mathring{\operatorname{div}}(-J)), \text{ and } \mathcal{H}_{\mathrm{N}} := \ker(\mathring{\operatorname{div}}) \cap \ker(\operatorname{div}(-J))$$

are the harmonic Dirichlet and Neumann fields in two spatial dimensions. Note that

$$\dim \mathcal{H}_{\mathrm{D}} = \dim \mathcal{H}_{\mathrm{N}} :$$

Indeed, for $q \in \mathrm{L}_2(\Omega)^2$, we have $q \in \mathcal{H}_{\mathrm{D}}$ if and only if $Jq \in \mathcal{H}_{\mathrm{N}}$. Thus, in order to prove (8) it remains to show $\mathcal{H}_{\mathrm{N}} = \{0\}$. For this, let $f \in \mathcal{H}_{\mathrm{N}}$. Note that without loss of generality, we may assume that f attains values in $\mathbb{R}$ only. Then $g(x+\mathrm{i}y) := f_1(x,y) - \mathrm{i} f_2(x,y)$ for all $(x,y) \in \Omega \subseteq \mathbb{R}^2 \cong \mathbb{C}$ defines a complex-valued function with $\operatorname{Re} g = f_1$ and $\operatorname{Im} g = -f_2$. It is not difficult to see that, locally as $f \in \mathcal{H}_{\mathrm{N}}$, g satisfies the Cauchy–Riemann equations in a distributional sense. In particular, $\operatorname{Re} g = f_1$ and $\operatorname{Im} g = -f_2$ are harmonic distributions and, by Weyl's lemma, $f_1, f_2 \in \mathrm{C}^\infty(\Omega)$. In consequence, g is holomorphic. Since Ω is simply connected, by Cauchy's integral theorem, there exists a potential G such that $G' = g$. Then, consider $F(x,y) := \operatorname{Re} G(x+\mathrm{i}y)$ and compute for $(x,y) \in \Omega$

$$\begin{aligned} \partial_x F(x,y) &= \operatorname{Re} G'(x+\mathrm{i}y) = \operatorname{Re} g(x+\mathrm{i}y) = f_1(x,y) \text{ and} \\ \partial_y F(x,y) &= \operatorname{Re} \mathrm{i} G'(x+\mathrm{i}y) = -\operatorname{Im} g(x+\mathrm{i}y) = f_2(x,y). \end{aligned}$$

Next, as Ω has the segment property, [Agm10, Theorem 3.2(2)] applies to an approximating sequence using the shift technique similar to the one considered in [Pic82, p. 170], and since $f \in L_2(\Omega)^2$, we infer that $F \in \operatorname{dom}(\operatorname{grad})$. By assumption, $\operatorname{grad} F = f \in \ker(\mathring{\operatorname{div}})$. In particular,

$$\langle \operatorname{grad} F, \operatorname{grad} F \rangle = -\langle F, \mathring{\operatorname{div}} \operatorname{grad} F \rangle = 0,$$

which implies that F is constant and, hence, $f = 0$ as desired.

As a consequence of the above decomposition, we also get $\operatorname{ran}(\operatorname{grad}) = \ker(\operatorname{div} J)$, $\operatorname{ran}(\mathring{\operatorname{grad}}) = \ker(\mathring{\operatorname{div}} J)$, $\ker(\operatorname{div}) = \operatorname{ran}(J \operatorname{grad})$ and $\ker(\mathring{\operatorname{div}}) = \operatorname{ran}(J \mathring{\operatorname{grad}})$. ♦

With this, we get yet another variant of Theorem 3.8.

Theorem 4.6. *Let $\Omega \subseteq \mathbb{R}^2$ be open, bounded and simply connected satisfying the segment property. For $0 < \alpha < \beta$, a sequence $(a_n)_{n \in \mathbb{N}}$ from $M(\alpha, \beta, \Omega)$ locally* H*-converges to an $a \in M(\alpha, \beta, \Omega)$ if and only if it $\tau(\operatorname{ran}(\mathring{\operatorname{grad}}), \operatorname{ran}(J \operatorname{grad}))$-converges to a.*

[2]Recall that the ranges are closed: Indeed, boundedness and the segment property of Ω yield that the Rellich–Kondrachov selection theorem holds (both $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}_2(\Omega)$ and $\mathring{\mathrm{H}}^1(\Omega) \hookrightarrow \mathrm{L}_2(\Omega)$ are compact embeddings). Thus, the claim follows by a standard argument for grad and $\mathring{\operatorname{grad}}$, see, e.g., again the FA-Toolbox in [PZ23], and the fact that J is a topological isomorphism.

Proof. The short proof of [Wau18, Theorem 5.11] only needs $\mathbb{R}^3$ to use the classical Helmholtz decomposition. The underlying [Wau18, Theorem 4.10] works on an abstract Hilbert space level. Hence, using Remark 4.5, we can readily rewrite [Wau18, Theorem 5.11] for $\mathbb{R}^2$. ❑

5. Examples

In this section, we will present a range of homogenisation examples (and their respective limits) that fall into the regime of Theorem 3.10. We focus on the pre-asymptotic homogenisation problems first and provide the respective limits after that. All of them can be written in the form

$$\big(\partial_{t,\nu} M_{0,n} + M_{1,n} + A\big)U = F, \tag{9}$$

for $M_{0,n}, M_{1,n} \in \mathcal{L}_{\mathrm{b}}(\mathcal{H}), n \in \mathbb{N}$, and suitable $\nu > 0$. In other words, the material laws are of the form $M_n(z) := M_{0,n} + z^{-1} M_{1,n}$ for $n \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \{0\}$ which readily implies $M_n \in \mathcal{M}(\mathcal{H}, \nu)$ for $n \in \mathbb{N}$ and for all $\nu > 0$. These examples will illustrate different convergence phenomena corresponding to the respective different decompositions $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 = \ker(A) \oplus \operatorname{ran}(A)$.

We state the examples next.

5.1. The Examples.

One of the standard situations of Theorem 3.10 is the case when $A = 0$, i.e., an ODE, in which case one can choose $\mathcal{H}_1 = \{0\}$ and $\mathcal{H}_0 = \mathcal{H} = \ker(A)$. This means only weak convergence of the solutions can be expected.

Example 5.1 (An ordinary differential equation)**.** Consider

$$\partial_{t,\nu} u_n(t,x) + \sin(2\pi n x)\, u_n(t,x) = f(t,x), \tag{10}$$

where $\Omega := (0,1)$, $\mathcal{H} := \mathrm{L}_2(\Omega)$, and $\nu > 2$.

Clearly, we have

$$\forall h \in \mathrm{L}_2(\Omega) : \operatorname{Re}\langle h, zh\rangle_{\mathrm{L}_2(\Omega)} + \operatorname{Re}\langle h, \sin(2\pi n\cdot) h\rangle_{\mathrm{L}_2(\Omega)} \geq \|h\|^2_{\mathrm{L}_2(\Omega)},$$

and

$$\forall h \in \mathrm{L}_2(\Omega) : \|h + z^{-1}\sin(2\pi n\cdot)h\|_{\mathrm{L}_2(\Omega)} \leq 2\|h\|_{\mathrm{L}_2(\Omega)}$$

for $\operatorname{Re} z > 2$ and $n \in \mathbb{N}$.

Thus, we are in the setting of Theorem 3.10 with $\nu_0 := 2$. In particular, (10) is well-posed (in the sense of Theorem 2.2 with $A := 0$) for each $n \in \mathbb{N}$, $\nu > \nu_0$, and $f \in \mathrm{L}_{2,\nu}(\mathbb{R}; \mathcal{H})$, and by [Wau16]

$$u_n(t,x) = \int_{-\infty}^{t} \mathrm{e}^{-(t-s)\sin(2\pi n x)} f(s,x)\,\mathrm{d}s \tag{11}$$

gives the solution sequence explicitly. ♦

The other extreme case is $\mathcal{H}_0 = \{0\}$, meaning A is one-to-one and the Hilbert space $\operatorname{dom}(A)$ compactly embeds into $\mathcal{H}_1 = \mathcal{H} = \operatorname{ran}(A)$. Hence, we obtain strong convergence of the solutions. In fact, a finite-dimensional kernel is already sufficient for strong convergence on the whole space since the weak and strong operator topology coincide on finite-dimensional spaces.

In the following we model oscillations with the indicator function $\mathbb{1}_{O_n} : \mathbb{R} \to \{0,1\}$ of the set

$$O_n := \bigcup_{k\in\mathbb{Z}} \big(\tfrac{2k}{2n}, \tfrac{2k+1}{2n}\big), n \in \mathbb{N},$$

see Figure 1. Note that $\mathbb{1}_{O_n}(x) = \mathbb{1}_{O_1}(nx)$ for $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

Example 5.2 (Finite-dimensional kernel)**.** This example appears in [FW18]. Let $n \in \mathbb{N}$ and set

$$\epsilon_n(x) := \mathbb{1}_{O_n}(x), \quad \sigma_n(x) := 1 - \mathbb{1}_{O_n}(x).$$

We consider the following rough-coefficient PDE[3]

$$\left[\partial_{t,\nu}\begin{pmatrix}\epsilon_n & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}\sigma_n & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & \partial_\#\\ \partial_\# & 0\end{pmatrix}\right] U_n = F, \tag{12}$$

where $\Omega := (0,1)$, $\mathcal{H} := \mathrm{L}_2(\Omega)^2$, $\nu > \varepsilon$ for an arbitrary $\varepsilon > 0$, $A := \left(\begin{smallmatrix}0 & \partial_\#\\ \partial_\# & 0\end{smallmatrix}\right)$, and $M_n(z) := \left(\begin{smallmatrix}\epsilon_n & 0\\ 0 & 1\end{smallmatrix}\right) + z^{-1}\left(\begin{smallmatrix}\sigma_n & 0\\ 0 & 0\end{smallmatrix}\right)$ for $n \in \mathbb{N}$ and $\operatorname{Re} z > \varepsilon$.

Obviously, $\operatorname{Re} zM_n(z) \geq \min\{1, \varepsilon\}$ and $\|M_n(z)\| \leq \max\{1, 1/\varepsilon\}$ hold for $\operatorname{Re} z > \varepsilon$ and $n \in \mathbb{N}$. Moreover, $\mathrm{H}^1_\#(0,1)$ compactly embeds into $\mathrm{L}_2(\Omega)$, by the Rellich–Kondrachov theorem.

Hence, we are in the setting of Theorem 3.10 with $\nu_0 := \varepsilon$, and $\mathcal{H}_0 = \ker(A)$ is the one-dimensional space of constant functions in $\mathrm{L}_2(\Omega)$ times itself. In particular, (12) is well-posed (in the sense of Theorem 2.2) for each $n \in \mathbb{N}$, $\nu > \nu_0$, and $F \in \mathrm{L}_{2,\nu}(\mathbb{R}; \mathcal{H})$. ♦

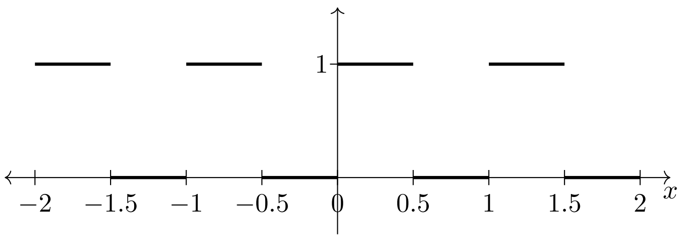


FIGURE 1. Graph of $\mathbb{1}_{O_n}$ for $n = 1$

The intermediate case, meaning both $\ker(A)$ and $\operatorname{ran}(A)$ are infinite-dimensional, has Maxwell's equations as a prominent example. We will additionally provide some other PDEs with such spatial derivative operators A.

Example 5.3 (Infinite-dimensional kernel and range)**.** We provide several examples in different spatial dimensions and with different spatial derivatives.

(i) The first example in one spatial dimension reads ($\mathbb{1}_{(-1,0)}$ is the indicator function of $(-1,0)$, ∂_x stands for the weak derivative on $\mathrm{L}_2(-1,1)$ and * denotes the L_2-adjoint operator)

$$\left[\partial_{t,\nu}\underbrace{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}_{=:M_{0,n}} + \underbrace{\begin{pmatrix}\sin(2\pi n\cdot) & 0\\ 0 & \sin(2\pi n\cdot)\end{pmatrix}}_{=:M_{1,n}} + \underbrace{\begin{pmatrix}0 & \partial_x\mathbb{1}_{(-1,0)}\\ -(\partial_x\mathbb{1}_{(-1,0)})^* & 0\end{pmatrix}}_{=:A}\right] U_n = F, \tag{13}$$

where $\Omega := (-1,1)$, $\mathcal{H} := \mathrm{L}_2(\Omega)^2$ and $\nu > 2$.

Note that the operator $\partial_x\mathbb{1}_{(-1,0)}$ on $\mathrm{L}_2(-1,1)$ can be decomposed in a block operator by decomposing $\mathrm{L}_2(-1,1)$ into $\mathrm{L}_2(-1,0) \oplus \mathrm{L}_2(0,1)$. Because of the Sobolev embedding theorem, the corresponding block operator is then

$$\partial_x\mathbb{1}_{(-1,0)} = \begin{pmatrix}\mathring{\partial}_{x,\{0\}} & 0\\ 0 & 0\end{pmatrix},$$

[3]For $\mathrm{C}^\infty_\#(0,1) := \{\varphi\restriction_{(0,1)} : \varphi \in \mathrm{C}^\infty(\mathbb{R}), \varphi(\cdot+1) = \varphi(\cdot)\}$, we define $\partial_x \restriction_{\mathrm{C}^\infty_\#} : \mathrm{C}^\infty_\#(0,1) \subseteq \mathrm{L}_2(0,1) \to \mathrm{L}_2(0,1)$ as the usual derivative. Weakly extending, we obtain $\partial_\# := -(\partial_x \restriction_{\mathrm{C}^\infty_\#})^*$ and $\partial_\# = -(\partial_\#)^* = \overline{\partial_x \restriction_{\mathrm{C}^\infty_\#}}$ with the domain as Hilbert space $\mathrm{H}^1_\#(0,1)$.

where $\mathring{\partial}_{x,\{0\}}$ is the derivative on $\mathrm{L}_2(-1,0)$ with domain $\{f \in \mathrm{H}^1(-1,0) \mid f(0) = 0\}$, i.e., the product of ∂_x and $\mathbb{1}_{(-1,0)}$ gives a "boundary" condition in the middle of the interval. Using a direct computation and integration by parts, the L_2-adjoint is then

$$-(\partial_x \mathbb{1}_{(-1,0)})^* = \begin{pmatrix} -(\mathring{\partial}_{x,\{0\}})^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathring{\partial}_{x,\{-1\}} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\mathring{\partial}_{x,\{-1\}}$ is the derivative on $\mathrm{L}_2(-1,0)$ with domain $\{f \in \mathrm{H}^1(-1,0) \mid f(-1) = 0\}$. This naturally induces a decomposition of $\mathrm{L}_2(-1,1)^2$ into

$$\underbrace{\mathrm{L}_2(-1,0)^2}_{=\mathcal{H}_1} \oplus \underbrace{\mathrm{L}_2(0,1)^2}_{=\mathcal{H}_0},$$

since we easily obtain $\mathrm{L}_2(-1,0)^2 = \operatorname{ran} A$ and $\mathrm{L}_2(0,1)^2 = \ker A$. Moreover, the Rellich–Kondrachov theorem implies that $\operatorname{dom}(A) \cap \mathcal{H}_1$ compactly embeds into $\mathcal{H}$.

Thus, similarly to Equation (10), we are in the setting of Theorem 3.10 with $\nu_0 \coloneqq 2$. In particular, (13) is well-posed (in the sense of Theorem 2.2) for each $n \in \mathbb{N}$, $\nu > \nu_0$, and $F \in \mathrm{L}_{2,\nu}(\mathbb{R}; \mathcal{H})$.

Going for two spatial dimensions we consider stratified coefficients, i.e., they oscillate only in one direction. For that purpose, we extend $\mathbb{1}_{O_n}$ (without changing the notation) to $\mathbb{1}_{O_n} : \mathbb{R}^2 \to \{0,1\}$ via $\mathbb{1}_{O_n}(x,y) \coloneqq \mathbb{1}_{O_n}(x)$ for all $x, y \in \mathbb{R}$.

(ii) For $\Omega \coloneqq (-2,2)^2$, $\Omega_1 \coloneqq (-1,1)^2$ and its indicator function $\mathbb{1}_{\Omega_1}$ and arbitrary constants $\epsilon_0, \mu_0 > 0$, consider the problem given by $\nu > \varepsilon$ for an arbitrary $\varepsilon > 0$ and the following operators on $\mathcal{H} \coloneqq \mathrm{L}_2(\Omega) \oplus \mathrm{L}_2(\Omega)^2$:

$$\begin{aligned} M_{0,n} &\coloneqq \mathbb{1}_{\Omega_1} \cdot \begin{pmatrix} 1 - \mathbb{1}_{O_n} & 0 \\ 0 & 1 + \mathbb{1}_{O_n} \end{pmatrix} + (1 - \mathbb{1}_{\Omega_1}) \cdot \begin{pmatrix} \epsilon_0 & 0 \\ 0 & \mu_0 \end{pmatrix}, \\ M_{1,n} &\coloneqq \mathbb{1}_{\Omega_1} \cdot \begin{pmatrix} \mathbb{1}_{O_n} & 0 \\ 0 & 0 \end{pmatrix}, \quad A \coloneqq \begin{pmatrix} 0 & \operatorname{div} \\ \mathring{\operatorname{grad}} & 0 \end{pmatrix}. \end{aligned} \tag{14}$$

Note that the oscillating coefficient is only active on the subdomain Ω_1. Thus, we have a non-periodic anisotropic homogenisation problem.

Obviously, $\operatorname{Re} z M_n(z) \geq \min\{\varepsilon\epsilon_0, \varepsilon\mu_0, 1, \varepsilon\}$ and $\|M_n(z)\| \leq \max\{\epsilon_0, \mu_0, 2, 1/\varepsilon\}$ hold for $n \in \mathbb{N}$ and $\operatorname{Re} z > \varepsilon$. By Remark 4.5, $\mathcal{H}$ decomposes into

$$\mathcal{H}_0 = \ker A = \begin{pmatrix} \{0\} \\ \operatorname{ran}(J \operatorname{grad}) \end{pmatrix} \quad \text{and} \quad \mathcal{H}_1 = \operatorname{ran} A = \begin{pmatrix} \mathrm{L}_2(\Omega) \\ \operatorname{ran}(\mathring{\operatorname{grad}}) \end{pmatrix}. \tag{15}$$

By the Rellich–Kondrachov theorem, $\mathring{\mathrm{H}}^1(\Omega) = \mathring{\mathrm{H}}^1(\Omega) \cap \operatorname{ran}(\operatorname{div}) = \mathring{\mathrm{H}}^1(\Omega) \cap \ker(\mathring{\operatorname{grad}})^\perp$ compactly embeds into $\mathrm{L}_2(\Omega)$. Going to the adjoints, we easily infer that $\mathrm{H}(\operatorname{div}, \Omega) \cap \operatorname{ran}(\mathring{\operatorname{grad}}) = \mathrm{H}(\operatorname{div}, \Omega) \cap \ker(\operatorname{div})^\perp$ also compactly embeds into $\mathrm{L}_2(\Omega)^2$, so $\operatorname{dom}(A) \cap \mathcal{H}_1$ compactly embeds into $\mathcal{H}$.

Alltogether, we are in the setting of Theorem 3.10 with $\nu_0 \coloneqq \varepsilon$. In particular, the PDE (9) arising from (14) is well-posed (in the sense of Theorem 2.2) for each $n \in \mathbb{N}$, $\nu > \nu_0$, and $F \in \mathrm{L}_{2,\nu}(\mathbb{R}; \mathcal{H})$.

(iii) The same as (ii) with slightly different $M_{0,n}$ and $M_{1,n}$:

$$\begin{aligned} M_{0,n} &\coloneqq \mathbb{1}_{\Omega_1} \cdot \begin{pmatrix} 1 + \mathbb{1}_{O_n} & 0 \\ 0 & 1 - \mathbb{1}_{O_n} \end{pmatrix} + (1 - \mathbb{1}_{\Omega_1}) \cdot \begin{pmatrix} \epsilon_0 & 0 \\ 0 & \mu_0 \end{pmatrix}, \\ M_{1,n} &\coloneqq \mathbb{1}_{\Omega_1} \cdot \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{O_n} \end{pmatrix}, \quad A \coloneqq \begin{pmatrix} 0 & \operatorname{div} \\ \mathring{\operatorname{grad}} & 0 \end{pmatrix}. \end{aligned}$$

In three dimensions, we also consider the stratified case, and we extend $\mathbb{1}_{O_n}$ once more (without changing the notation) to $\mathbb{1}_{O_n}\colon \mathbb{R}^3 \to \{0,1\}$ via $\mathbb{1}_{O_n}(x,y,z) \coloneqq \mathbb{1}_{O_n}(x)$ for all $x, y, z \in \mathbb{R}$.

(iv) The fourth and final example are Maxwell's equations (cf. Example 2.4). For $\Omega \coloneqq (-2,2)^3$, $\Omega_1 \coloneqq (-1,1)^3$ and its indicator function $\mathbb{1}_{\Omega_1}$ and arbitrary constants $\epsilon, \mu, \sigma, \epsilon_0, \mu_0 > 0$, consider the problem given by $\nu > \varepsilon$ for an arbitrary $\varepsilon > 0$ and the following operators on $\mathcal{H} \coloneqq \mathrm{L}_2(\Omega)^3 \oplus \mathrm{L}_2(\Omega)^3$:

$$\begin{aligned} M_{0,n} &\coloneqq \mathbb{1}_{\Omega_1} \cdot \begin{pmatrix} \epsilon(1-\mathbb{1}_{O_n}) & 0 \\ 0 & \mu(1+\mathbb{1}_{O_n}) \end{pmatrix} + (1-\mathbb{1}_{\Omega_1}) \cdot \begin{pmatrix} \epsilon_0 & 0 \\ 0 & \mu_0 \end{pmatrix}, \\ M_{1,n} &\coloneqq \mathbb{1}_{\Omega_1} \cdot \begin{pmatrix} \sigma \mathbb{1}_{O_n} & 0 \\ 0 & 0 \end{pmatrix}, \quad A \coloneqq \begin{pmatrix} 0 & -\operatorname{curl} \\ \mathring{\operatorname{curl}} & 0 \end{pmatrix}. \end{aligned} \tag{16}$$

For $n \in \mathbb{N}$ and $\operatorname{Re} z > \varepsilon$, we have $\operatorname{Re} z M_n(z) \geq \min(\varepsilon\epsilon_0, \varepsilon\mu_0, \sigma, \varepsilon\epsilon, \varepsilon\mu)$ and $\|M_n(z)\| \leq \max(\epsilon_0, \mu_0, 2\mu, \sigma/\varepsilon, \epsilon)$. By Remark 3.7, the kernel and range of A yield the following decomposition of $\mathcal{H}$:

$$\mathcal{H}_0 = \begin{pmatrix} \ker(\mathring{\operatorname{curl}}) \\ \ker(\operatorname{curl}) \end{pmatrix} = \begin{pmatrix} \operatorname{ran}(\mathring{\operatorname{grad}}) \\ \operatorname{ran}(\operatorname{grad}) \end{pmatrix} \quad \text{and} \quad \mathcal{H}_1 = \begin{pmatrix} \operatorname{ran}(\operatorname{curl}) \\ \operatorname{ran}(\mathring{\operatorname{curl}}) \end{pmatrix}. \tag{17}$$

Moreover, we obtain that $\mathring{\mathrm{H}}(\operatorname{curl},\Omega) \cap \operatorname{ran}(\operatorname{curl}) = \mathring{\mathrm{H}}(\operatorname{curl},\Omega) \cap \ker(\operatorname{div})$ and $\mathrm{H}(\operatorname{curl},\Omega) \cap \operatorname{ran}(\mathring{\operatorname{curl}}) = \mathrm{H}(\operatorname{curl},\Omega) \cap \ker(\mathring{\operatorname{div}})$ are each compactly embedded into $\mathrm{L}_2(\Omega)^3$ (Picard–Weber–Weck selection theorem; see [Pic84]).

Therefore, we are in the setting of Theorem 3.10 with $\nu_0 \coloneqq \varepsilon$. In particular, the PDE (9) arising from (16) is well-posed (in the sense of Theorem 2.2) for each $n \in \mathbb{N}$, $\nu > \nu_0$, and $F \in \mathrm{L}_{2,\nu}(\mathbb{R}; \mathcal{H})$. ♦

5.2. **Homogenisation limits of the examples.** In this section we calculate the homogenisation limits of the previous examples.

Example 5.4 (An ordinary differential equation)**.** For Example 5.1, the convergence of material laws necessary for Theorem 3.10 boils down to finding an $M\colon \mathbb{C}_{\mathrm{Re}>\nu_0} \to \mathcal{L}_{\mathrm{b}}(\mathcal{H})$ with $\|M(z)\| \leq 2$ and $M(z)^{-1} \in \mathcal{L}_{\mathrm{b}}(\mathcal{H})$ on $\mathbb{C}_{\mathrm{Re}>\nu_0}$ such that $M_n(z)^{-1}$ converges in the weak operator topology to $M(z)^{-1}$ for each $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$, where, for $n \in \mathbb{N}$ and $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$, the operator $M_n(z) \in \mathcal{L}_{\mathrm{b}}(\mathcal{H})$ stands for the multiplication-with-$(1+z^{-1}\sin(2\pi n \cdot))$ operator. Then, the well-posed (in the sense of Theorem 2.2 with $A \coloneqq 0$) limit problem is given by

$$\partial_{t,\nu} M(\partial_{t,\nu}) u^{\mathrm{hom}}(t,x) = f(t,x),$$

for each $\nu > \nu_0$ and $f \in \mathrm{L}_{2,\nu}(\mathbb{R}; \mathcal{H})$.

By [Wau16], we have[4]

$$M(z) = 1 + \sum_{j=1}^{\infty} \left(-\sum_{m=1}^{\infty} \frac{(2m)!}{(2^m m!)^2} z^{-2m} \right)^j \tag{18}$$

for $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$.

Therefore, the homogenised problem reads

$$\partial_{t,\nu} u^{\mathrm{hom}}(t,x) + \sum_{j=1}^{\infty} \partial_{t,\nu} \left(-\sum_{m=1}^{\infty} \frac{(2m)!}{(2^m m!)^2} \partial_{t,\nu}^{-2m} \right)^j u^{\mathrm{hom}}(t,x) = f(t,x),$$

[4]After some tedious but basic calculations, one gets that the operator norm of the double series is strictly smaller than 1. Thus, one indeed obtains $M(z), M(z)^{-1} \in \mathcal{L}_{\mathrm{b}}(\mathcal{H})$ with $\|M(z)\| \leq 2$ on $\mathbb{C}_{\mathrm{Re}>\nu_0}$.

Strictly speaking, [Wau16] directly shows convergence of the solution operators. Convergence of the corresponding material laws follows by compactness, see, e.g., [BSW23, Theorem 5.6 and Lemma 5.7], and the unique correspondence between material laws and their respective operators.

for each $\nu > \nu_0$ and $f \in \mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H})$. Moreover by [Wau16], the homogenised solution explicitly reads

$$u^{\mathrm{hom}}(t,x) = \int_{-\infty}^{t} I_0(t-s)f(s,x)\,\mathrm{d}s,$$

where I_0 is the modified Bessel function of first kind. Comparing this to (11) and consulting Theorem 3.10, we conclude that u_n converges to u^{hom} weakly but (due to the oscillation) in general not strongly for each $\nu > \nu_0$ and $f \in \mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H})$. ♦

Example 5.5 (Finite-dimensional kernel)**.** For Example 5.2, we will manually calculate the limit of the material laws that Theorem 3.10 asks for. The orthogonal projection from $\mathrm{L}_2(0,1)$ onto the closed subspace of constant functions is given by the integral mean. Recalling the definition of the nonlocal H-topology and considering $(\epsilon_n)_{n\in\mathbb{N}}$ first, we need to find the limits of $(\int_0^1 \epsilon_n(x)\,\mathrm{d}x)^{-1}$, $\int_0^1 \varphi(x)\epsilon_n(x)\,\mathrm{d}x$, and $\int_0^1 \varphi(x)\epsilon_n(x)\psi(x)\,\mathrm{d}x$ for all $\varphi,\psi \in \mathrm{L}_2(0,1)$ with integral mean 0. By Theorem A.3 and since $\int_0^1 \epsilon_1(x)\,\mathrm{d}x = 1/2$, the respective limits are 2, 0, and $\int_0^1 \varphi(x)(1/2)\psi(x)\,\mathrm{d}x$ respectively. With that, one easily proves $\epsilon_n \to 1/2$ in the nonlocal H-topology. Arguing similarly for $(\sigma_n)_{n\in\mathbb{N}}$, we obtain the homogenised material law $M(z) := \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} + z^{-1}\begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}(\ker(A),\operatorname{ran}(A))$ for $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$.

Therefore, Theorem 3.10 yields the well-posed homogenised problem

$$\left[\partial_{t,\nu}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \partial_{\#} \\ \partial_{\#} & 0 \end{pmatrix}\right] U^{\mathrm{hom}} = F,$$

and the solutions U_n converge strongly to U^{hom} for each $\nu > \nu_0$ and $F \in \mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H})$. ♦

Example 5.6 (Infinite-dimensional kernel and range)**.** Let us look at the PDE-sequences in Example 5.3 and compute their limits.

(i) Example 5.3 (i) is in some sense a concatenation of two problems. We have already discussed that the decomposition of $\mathcal{H}$ into the range $\mathcal{H}_1$ and kernel $\mathcal{H}_0$ of A is given by $\mathrm{L}_2(-1,1)^2 = \mathrm{L}_2(-1,0)^2 \oplus \mathrm{L}_2(0,1)^2$. Hence, we can decompose U, F, A, etc. accordingly and obtain the following representation of the PDE for $n \in \mathbb{N}$, $\nu > \nu_0$ and $F \in \mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H})$

$$\left[\partial_{t,\nu}\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} + \sin(2\pi n\cdot)\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}\right.$$
$$\left. + \begin{pmatrix} \begin{pmatrix} 0 & \mathring{\partial}_{x,\{0\}} \\ \mathring{\partial}_{x,\{-1\}} & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}\right] \begin{pmatrix} U_{n,1} \\ U_{n,2} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}.$$

We now manually calculate the homogenised limit of the material laws.

Because of the diagonal shape of the material law the nonlocal H-convergence is decoupled. To be precise, we only have to deal with $a_{n,00}^{-1}$ and $a_{n,11}$ (Note that we have swapped the order of $\operatorname{ran} A$ and $\ker A$ in our decomposition, hence $a_{n,00}$ is in the lower right corner and $a_{n,11}$ in the upper left).

The limiting process on $\mathcal{H}_1$ is a direct consequence of Theorem A.3 implying $a_{n,11} = 1 + \frac{1}{z}\sin(2\pi n\cdot) \to 1$ in the weak operator topology for every $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$. Thus, on $(-1,0)$, we obtain the well-posed homogenised problem

$$\left[\partial_{t,\nu}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \mathring{\partial}_{x,\{0\}} \\ \mathring{\partial}_{x,\{-1\}} & 0 \end{pmatrix}\right] U_1^{\mathrm{hom}} = F_1,$$

for each $\nu > \nu_0$ and $F_1 \in \mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H}_1)$.

On $\mathcal{H}_0$, we are in the setting of Example 5.1. Therefore, on $(0,1)$, we obtain from Example 5.4 (omitting the system) the homogenised solution

$$U_2^{\mathrm{hom}}(t,x) = \int_0^t I_0(t-s)F_2(s,x)\,\mathrm{d}s,$$

for each $\nu > \nu_0$ and $F_2 \in \mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H}_0)$.

To sum up, the homogenised material law $M(z) \in \mathcal{M}(\mathcal{H}_0,\mathcal{H}_1)$ for $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$ is given by $\left(\begin{smallmatrix} M_{11} & 0 \\ 0 & M_{00}(z)\end{smallmatrix}\right)$, where $M_{11}(z) := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right)$ and $M_{00}(z)$ is (18) times $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right)$. Theorem 3.10 yields well-posedness of the corresponding homogenised problem

$$\left[\partial_{t,\nu}\begin{pmatrix} M_{11} & 0 \\ 0 & M_{00}(\partial_{t,\nu})\end{pmatrix} + A\right]\begin{pmatrix} U_1^{\mathrm{hom}} \\ U_2^{\mathrm{hom}}\end{pmatrix} = F,$$

strong convergence of $U_{n,1}$ to U_1^{hom} and weak convergence of $U_{n,2}$ to U_2^{hom} for each $\nu > \nu_0$ and $F \in \mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H})$. Recall from Example 5.4 that in general U_2^{hom} is indeed only reached in a week sense.

(ii) In Example 5.3 (ii) the block operators were given with respect to the decomposition $\mathrm{L}_2(\Omega) \oplus \mathrm{L}_2(\Omega)^2$. On $\Omega \setminus \overline{\Omega_1}$, the material law is constantly $\left(\begin{smallmatrix} \epsilon_0 & 0 \\ 0 & \mu_0\end{smallmatrix}\right)$.

The $\ker A \oplus \operatorname{ran} A$ decomposition (15) lets us compute the nonlocal H-limit in the first row manually. With Theorem A.3, we infer

$$1 - \mathbb{1}_{O_n} + \frac{1}{z}\mathbb{1}_{O_n} \to \frac{1}{2} + \frac{1}{z}\frac{1}{2}$$

for $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$ in $\mathcal{L}_{\mathrm{b}}(\mathrm{L}_2(\Omega_1))$ in the weak operator topology. That implies

$$\mathbb{1}_{\Omega_1}\left(1 - \mathbb{1}_{O_n} + \frac{1}{z}\mathbb{1}_{O_n}\right) + (1-\mathbb{1}_{\Omega_1})\epsilon_0 \to \mathbb{1}_{\Omega_1}\left(\frac{1}{2} + \frac{1}{z}\frac{1}{2}\right) + (1-\mathbb{1}_{\Omega_1})\epsilon_0$$

for $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$ in $\mathcal{L}_{\mathrm{b}}(\mathrm{L}_2(\Omega))$ in the weak operator topology.

The second row of (15) asks for the $\tau(\operatorname{ran}(J\operatorname{grad}), \operatorname{ran}(\mathring{\operatorname{grad}}))$-limit of

$$\mathbb{1}_{\Omega_1}\begin{pmatrix} 1+\mathbb{1}_{O_n} & 0 \\ 0 & 1+\mathbb{1}_{O_n}\end{pmatrix} + (1-\mathbb{1}_{\Omega_1})\mu_0.$$

We compute the $\tau(\operatorname{ran}(\mathring{\operatorname{grad}}), \operatorname{ran}(J\operatorname{grad}))$-limit

$$\mathbb{1}_{\Omega_1}\begin{pmatrix} (1+\mathbb{1}_{O_n})^{-1} & 0 \\ 0 & (1+\mathbb{1}_{O_n})^{-1}\end{pmatrix} + (1-\mathbb{1}_{\Omega_1})\mu_0^{-1} \to \mathbb{1}_{\Omega_1}\begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{3}{4}\end{pmatrix} + (1-\mathbb{1}_{\Omega_1})\mu_0^{-1},$$

and via inverting the $\tau(\operatorname{ran}(J\operatorname{grad}), \operatorname{ran}(\mathring{\operatorname{grad}}))$-limit

$$\mathbb{1}_{\Omega_1}\begin{pmatrix} 1+\mathbb{1}_{O_n} & 0 \\ 0 & 1+\mathbb{1}_{O_n}\end{pmatrix} + (1-\mathbb{1}_{\Omega_1})\mu_0 \to \mathbb{1}_{\Omega_1}\begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{4}{3}\end{pmatrix} + (1-\mathbb{1}_{\Omega_1})\mu_0,$$

using Theorem 4.1, Theorem 4.6, Theorem 4.2 and Theorem 3.3.

All in all, the homogenised material law $M(z) \in \mathcal{M}(\mathcal{H}_0,\mathcal{H}_1)$ for $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$ is given by $M(z) := M_0 + z^{-1}M_1$, where

$$M_0 := \mathbb{1}_{\Omega_1}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{4}{3}\end{pmatrix}\end{pmatrix} + (1-\mathbb{1}_{\Omega_1})\begin{pmatrix} \epsilon_0 & 0 \\ 0 & \mu_0\end{pmatrix}, \quad M_1 := \mathbb{1}_{\Omega_1}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0\end{pmatrix}.$$

Theorem 3.10 yields well-posedness of the homogenised problem

$$(\partial_{t,\nu}M_0 + M_1 + A)U^{\mathrm{hom}} = F$$

with weak convergence of the $\mathrm{L}_{2,\nu}(\mathbb{R};\ker(A))$-component of the solution sequence and strong convergence of the $\mathrm{L}_{2,\nu}(\mathbb{R};\operatorname{ran}(A))$-component of the solution sequence for each $\nu > \nu_0$ and $F \in \mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H})$.

(iii) For Example 5.3 (iii), we exactly repeat the ideas from (ii).

The first row of (15) results in computing

$$\mathbb{1}_{\Omega_1}(1+\mathbb{1}_{O_n}) + (1-\mathbb{1}_{\Omega_1})\epsilon_0 \to \mathbb{1}_{\Omega_1}\frac{3}{2} + (1-\mathbb{1}_{\Omega_1})\epsilon_0$$

in $\mathcal{L}_{\mathrm{b}}(\mathrm{L}_2(\Omega))$ in the weak operator topology.

For the second row of (15), we have

$$\begin{aligned}\mathbb{1}_{\Omega_1}\big(1-(1-\tfrac{1}{z})\mathbb{1}_{O_n}\big)^{-1}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + (1-\mathbb{1}_{\Omega_1})\mu_0^{-1}&\\ \to \mathbb{1}_{\Omega_1}\begin{pmatrix}\frac{2}{1+\frac{1}{z}} & 0\\ 0 & \frac{1}{2}(1+z)\end{pmatrix} + (1-\mathbb{1}_{\Omega_1})\mu_0^{-1}&\end{aligned}$$

in $\tau(\operatorname{ran}(\mathring{\operatorname{grad}}), \operatorname{ran}(J\operatorname{grad}))$ for $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$. Hence, the $\tau(\operatorname{ran}(J\operatorname{grad}), \operatorname{ran}(\mathring{\operatorname{grad}}))$-limit reads

$$\mathbb{1}_{\Omega_1}\begin{pmatrix}\frac{1}{2}+\frac{1}{z}\frac{1}{2} & 0\\ 0 & 2(1+z)^{-1}\end{pmatrix} + (1-\mathbb{1}_{\Omega_1})\mu_0$$

for $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$. If we substitute

$$2(1+z)^{-1} = \frac{1}{z}(z+1-1)2(1+z)^{-1} = \frac{1}{z}(2-2(1+z)^{-1}),$$

the homogenised material law $M(z) \in \mathcal{M}(\mathcal{H}_0,\mathcal{H}_1)$, $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$, is given by $M(z) := M_0 + z^{-1}M_1$, where

$$\begin{aligned} M_0 &:= \mathbb{1}_{\Omega_1}\begin{pmatrix}\frac{3}{2} & 0\\ 0 & \begin{pmatrix}\frac{1}{2} & 0\\ 0 & 0\end{pmatrix}\end{pmatrix} + (1-\mathbb{1}_{\Omega_1})\begin{pmatrix}\epsilon_0 & 0\\ 0 & \mu_0\end{pmatrix},\\ M_1 &:= \mathbb{1}_{\Omega_1}\begin{pmatrix}0 & 0\\ 0 & \begin{pmatrix}\frac{1}{2} & 0\\ 0 & 2-2(1+z)^{-1}\end{pmatrix}\end{pmatrix}. \end{aligned} \tag{19}$$

Note that by the limit process, we obtained an operator with the memory term $(1+z)^{-1} \approx (1+\partial_{t,\nu})^{-1}$.

(iv) For Example 5.3 (iv), we have to consider the decomposition (17).

For the first row, we need to find the $\tau(\operatorname{ran}(\mathring{\operatorname{grad}}), \operatorname{ran}(\operatorname{curl}))$-limit of

$$\mathbb{1}_{\Omega_1}\begin{pmatrix}(1-\mathbb{1}_{O_n})\epsilon+\frac{1}{z}\mathbb{1}_{O_n}\sigma & 0 & 0\\ 0 & (1-\mathbb{1}_{O_n})\epsilon+\frac{1}{z}\mathbb{1}_{O_n}\sigma & 0\\ 0 & 0 & (1-\mathbb{1}_{O_n})\epsilon+\frac{1}{z}\mathbb{1}_{O_n}\sigma\end{pmatrix} + (1-\mathbb{1}_{\Omega_1})\epsilon_0$$

for $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$. Applying Theorem 3.8, Theorem 4.2 and Theorem 3.3, we obtain the limit

$$\mathbb{1}_{\Omega_1}\begin{pmatrix}\frac{1}{z}2(\epsilon^{-1}\frac{1}{z}+\sigma^{-1})^{-1} & 0 & 0\\ 0 & \frac{1}{2}\epsilon+\frac{1}{z}\frac{1}{2}\sigma & 0\\ 0 & 0 & \frac{1}{2}\epsilon+\frac{1}{z}\frac{1}{2}\sigma\end{pmatrix} + (1-\mathbb{1}_{\Omega_1})\epsilon_0$$

for $z \in \mathbb{C}_{\mathrm{Re}>\nu_0}$.

For the second row of (17), we need to find the $\tau(\operatorname{ran}(\operatorname{grad}), \operatorname{ran}(\overset{\circ}{\operatorname{curl}}))$-limit of

$$\mathbb{1}_{\Omega_1}\begin{pmatrix}(1+\mathbb{1}_{O_n})\mu & 0 & 0\\ 0 & (1+\mathbb{1}_{O_n})\mu & 0\\ 0 & 0 & (1+\mathbb{1}_{O_n})\mu\end{pmatrix} + (1-\mathbb{1}_{\Omega_1})\mu_0.$$

Theorem 4.4, Theorem 4.2 and Theorem 3.3 yield the limit

$$\mathbb{1}_{\Omega_1}\begin{pmatrix}\frac{4}{3}\mu & 0 & 0\\ 0 & \frac{3}{2}\mu & 0\\ 0 & 0 & \frac{3}{2}\mu\end{pmatrix} + (1-\mathbb{1}_{\Omega_1})\mu_0.$$

Thus, if we substitute

$$\begin{aligned}\frac{1}{z}2\Big(\epsilon^{-1}\frac{1}{z}+\sigma^{-1}\Big)^{-1} &= \frac{1}{z}2\epsilon z(\sigma+z\epsilon)^{-1}\sigma = \frac{1}{z}2(\epsilon z+\sigma-\sigma)(\sigma+z\epsilon)^{-1}\sigma\\ &= \frac{1}{z}2\big(1-\sigma(\sigma+z\epsilon)^{-1}\big)\sigma,\end{aligned}$$

the homogenised material law $M(z) \in \mathcal{M}(\mathcal{H}_0, \mathcal{H}_1)$ is given by $M(z) := M_0 + z^{-1}M_1$, where

$$M_0 := \mathbb{1}_{\Omega_1}\cdot\begin{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & \frac{1}{2}\epsilon & 0\\ 0 & 0 & \frac{1}{2}\epsilon\end{pmatrix} & 0\\ 0 & \begin{pmatrix}\frac{4}{3}\mu & 0 & 0\\ 0 & \frac{3}{2}\mu & 0\\ 0 & 0 & \frac{3}{2}\mu\end{pmatrix}\end{pmatrix} + (1-\mathbb{1}_{\Omega_1})\cdot\begin{pmatrix}\epsilon_0 & 0\\ 0 & \mu_0\end{pmatrix},$$

$$M_1 := \mathbb{1}_{\Omega_1}\cdot\begin{pmatrix}\begin{pmatrix}2\sigma(1-\sigma(\sigma+z\epsilon)^{-1}) & 0 & 0\\ 0 & \frac{\sigma}{2} & 0\\ 0 & 0 & \frac{\sigma}{2}\end{pmatrix} & 0\\ 0 & 0\end{pmatrix}$$

As in (iii), this material law has a memory term. This time, it is $(\sigma + z\epsilon)^{-1} \approx (\sigma + \partial_{t,\nu}\epsilon)^{-1}$. Theorem 3.10 yields well-posedness of the homogenised problem

$$\big(\partial_{t,\nu}M_0 + M_1 + A\big)U^{\mathrm{hom}} = F$$

with weak convergence of the $\mathrm{L}_{2,\nu}(\mathbb{R};\ker(A))$-component of the solution sequence and strong convergence of the $\mathrm{L}_{2,\nu}(\mathbb{R};\operatorname{ran}(A))$-component of the solution sequence for each $\nu > \nu_0$ and $F \in \mathrm{L}_{2,\nu}(\mathbb{R};\mathcal{H})$. ♦

6. Numerical Simulations

This section is devoted to complement a numerical study for our theoretical findings from the previous sections. The computations presented here will support our results concerning strong and weak convergence. The simulations are carried out in $\mathbb{SOFE}$, a finite-element framework in Matlab and Octave, see `github.com/SOFE-Developers/SOFE`.

For the temporal dimension in the examples to come, we use a discontinuous Galerkin method in time, whereas a continuous Galerkin method is applied for the spatial directions. The details of this method tailored for evolutionary equations are provided in [FTW19]; we briefly summarise the essential parts next. For a fixed $T > 0$ and the corresponding time interval $[0, T]$, consider a partition $0 = t_0 < \cdots < t_M = T$ for some $M \in \mathbb{N}$ and set $I_m := (t_{m-1}, t_m]$ for $1 \le m \le M$. For $\mathcal{U}$, a piecewise polynomial discontinuous space of degree 1 in time corresponding to the partition of $[0, T]$ and an A-conforming piecewise polynomial space of degree 1 in space corresponding to the spatial Hilbert space $\mathcal{H}$, the method is now given by: Find $U \in \mathcal{U}$ such that for all $m \in \{1, \ldots, M\}$ and $\Phi \in \mathcal{U}$

$$Q_m\left[(\partial_t M_0 + M_1 + A)U, \Phi\right]_\rho + \langle M_0[\![U]\!]_{m-1}, \Phi^+_{m-1}\rangle_{\mathcal{H}} = Q_m\left[F, \Phi\right]_\rho + \langle M_0U_0, \Phi(0+)\rangle_{\mathcal{H}}.$$

Here, $Q_m\,[a,b]_\rho \coloneqq Q_m\,[\langle a,b\rangle_{\mathcal{H}}]$ is the weighted right-sided Gauß–Radau quadrature rule with the property

$$Q_m\,[p] = \int_{I_m} p(t)\mathrm{e}^{-2\rho(t-t_{m-1})}\,\mathrm{d}t \quad \text{for all} \quad p \in \mathcal{P}_2(I_m),$$

i.e., it approximates the exponentially weighted scalar product in space-time. Furthermore, we denote by

$$[\![U]\!]_{m-1} \coloneqq \begin{cases} U(t_{m-1}+) - U(t_{m-1}-), & m \in \{2,\dots,M\} \\ U(t_0+), & m = 1, \end{cases}$$

the jump of U at t_{m-1} and set $\Phi^+_{m-1} \coloneqq \Phi(t_{m-1}+)$.

For convenience, we restricted our attention to finite intervals as time-horizon instead of $\mathbb{R}$ that was used to model the temporal scale in our theoretical sections.

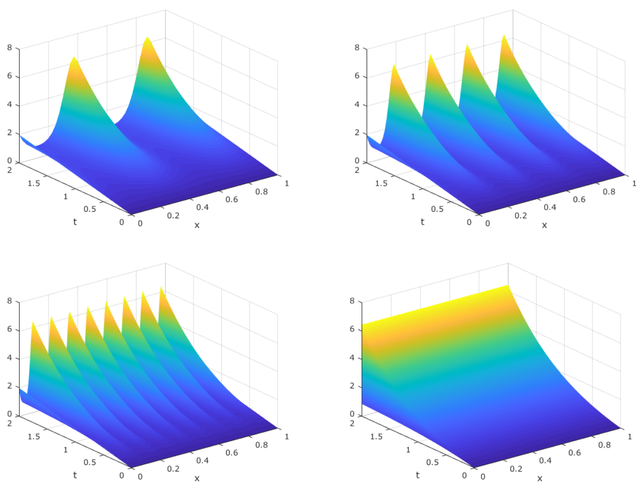
6.1. **Continuation of Example 5.1.** We consider the sequence of equations ($n \in \mathbb{N}$)

$$\partial_t u_n(t,x) + \sin(2\pi n x)\,u_n(t,x) = f(t,x)$$

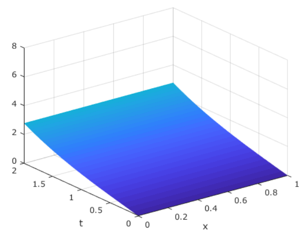
in the domain $[0,2]\times[0,1]$. Asking for initial zero conditions, we find the solution of the homogenised problem

$$u^{\mathrm{hom}}(t,x) = \int_0^t I_0(t-s)f(s,x)\,\mathrm{d}s,$$

where I_0 is the modified Bessel function of first kind (see Example 5.4 and [Wau16]). For our numerical experiment we choose $f(t,x)=1$ in $[0,2]\times[0,1]$. Figure 2 shows the



(A) Solutions of Example 5.1 for $n \in \{2,4,8,1024\}$



(B) Solutions of the homgenisation limit of Example 5.1

FIGURE 2. Solutions of Example 5.1 for $n \in \{2,4,8,1024\}$ and for the homogenised problem

solutions of Example 5.1 for different values of n. We observe that doubling n doubles the number of waves in the solution. Thus, we have a shrinking effect of a periodic function in x-direction. In contrast to that, the last plot shows u^{hom}, the solution of the homogenised problem.

This visualises weak but the lack of strong convergence. Unfortunately, we cannot compute the quantity

$$\sup_{v \in \mathrm{L}_2([0,2]\times[0,1]), \|v\| \leq 1} |\langle u_n - u^{\mathrm{hom}}, v\rangle_{\mathrm{L}_2}|,$$

and thus approximate it with the following small sample of example functions

$$v \in \big\{(t,x) \mapsto 1, (t,x) \mapsto x, (t,x) \mapsto x^2, (t,x) \mapsto \sin(\pi x), (t,x) \mapsto t\big\}.$$

The first and last function are constant in space, while all but the last function are constant in time.

Figure 3 shows the results of these scalar products. The value of the homogenised

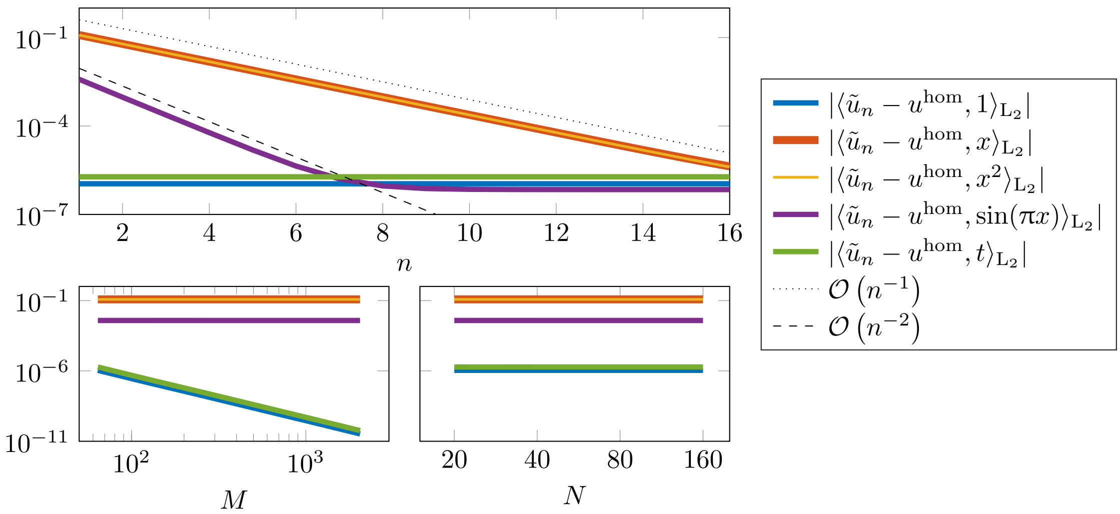


FIGURE 3. Results investigating weak convergence for Example 5.1

problem was calculated using MAPLE and the given formula for u^{hom}. For the numerical simulation we used a fixed equidistant mesh with $N = 10 \cdot n$ cells in the spatial direction and $M = 64$ cells in the time direction in the upper plot. We do observe, that the functions v being constant in space give a constant result, while we have results of order $\mathcal{O}(n^{-1})$ for $v(t,x) = x$ and $v(t,x) = x^2$. For $v(t,x) = \sin(\pi x)$, we observe an initial second order convergence $\mathcal{O}(n^{-2})$ but the graph stagnates at a certain value. The reason for this behaviour lies in the approximation of u^{hom}. In fact, Figure 3 does not show $\langle u_n - u^{\mathrm{hom}}, v\rangle_{\mathrm{L}_2}$ but rather

$$\langle \tilde{u}_n - u^{\mathrm{hom}}, v\rangle_{\mathrm{L}_2} = \langle \tilde{u}_n - u_n, v\rangle_{\mathrm{L}_2} + \langle u_n - u^{\mathrm{hom}}, v\rangle_{\mathrm{L}_2}$$

replacing u_n by a numerical approximation $\tilde{u}_n$. In order to investigate the effect of this numerical approximation, we fixed $n = 1$ for the lower plots in Figure 3 and varied the mesh in time with $M \in \{64, 128, 256, 512, 1024, 2048\}$ and in space with $N \in \{20, 40, 80, 160\}$, respectively. We observe a reduction for $v = 1$ and $v(t,x) = t$, when refining in time, but none for refining in space. This indicates, that the accuracy is limited by the approximation errors in time, which are of order 10^{-6} in this example.

These plots underpin the weak convergence $u_n \rightharpoonup u^{\mathrm{hom}}$, and even suggest a rate of order n^{-1}. If true in general, this would complement quantitative findings for strong convergence for evolutionary equations (cf. [CEW24, FW18]).

6.2. **Continuation of Example 5.3 (i).** We consider the sequence of equations ($n \in \mathbb{N}$)

$$\left[\partial_t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \sin(2\pi n x) & 0 \\ 0 & \sin(2\pi n x) \end{pmatrix} + \begin{pmatrix} 0 & \partial_x \mathbb{1}_{[-1,0]} \\ -(\partial_x \mathbb{1}_{[-1,0]})^* & 0 \end{pmatrix}\right] \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} \sin(2\pi t) \\ x - \frac{1}{2} \end{pmatrix}$$

in $[0,2] \times [-1,1]$. Figure 4 shows u_n in the upper and v_n in the lower subplots for different

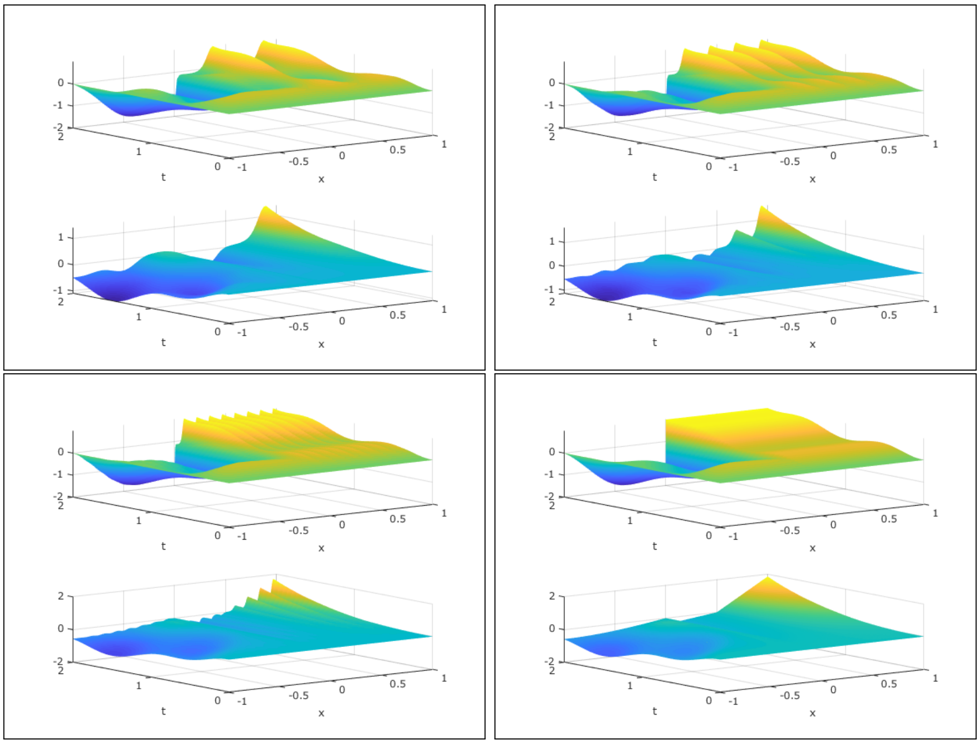


FIGURE 4. Solutions of Example 5.3 (i) for $n \in \{2, 4, 8, 1024\}$ from top left to bottom right.

$n \in \mathbb{N}$. As expected, they suggest strong convergence on the spatial $[0,1]$-part and weak convergence on the $[-1,0]$-part. In order to investigate this further, we use a numerical solution $(\tilde{u}^{\text{hom}}, \tilde{v}^{\text{hom}})$ by setting $n = 2^{13}$, as well as mesh parameters $N = 40n = 327\,680$, $M = 2^8 = 256$, and we increase the degrees of the piecewise polynomials to 3 in space and 2 in time.

For the numerical simulations, we again vary n and chose $M = 2^6$, $N = 40 \cdot n$ and polynomial degrees 2 in space and 1 in time. Figure 5 shows the results of the simulations for u in the upper and v in the lower graphs. Subject to (small) approximation errors elaborated on in the previous example, both components of the solution weakly converge. The plots suggest convergence of order $\mathcal{O}\left(n^{-1}\right)$ for either component.

6.3. **Continuation of Example 5.3 (ii).** We consider the sequence of equations ($n \in \mathbb{N}$)

$$(\partial_t M_{0,n} + M_{1,n} + A) U_n = \begin{pmatrix} \sin(2\pi t) \\ 0 \end{pmatrix}$$

with a final time horizon $T = 2$.

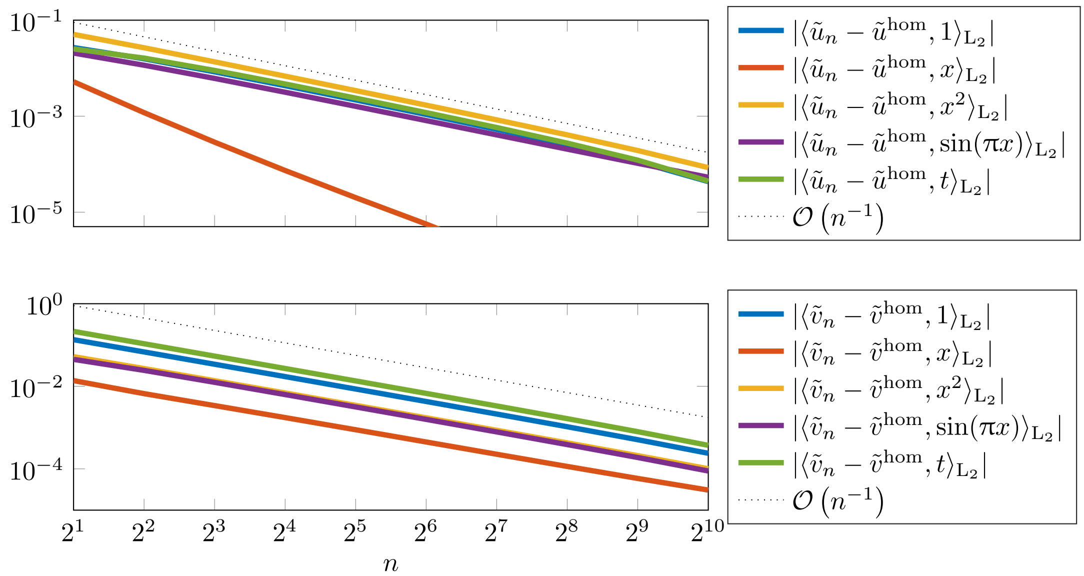


FIGURE 5. Results investigating weak convergence for Example 5.3 (i)

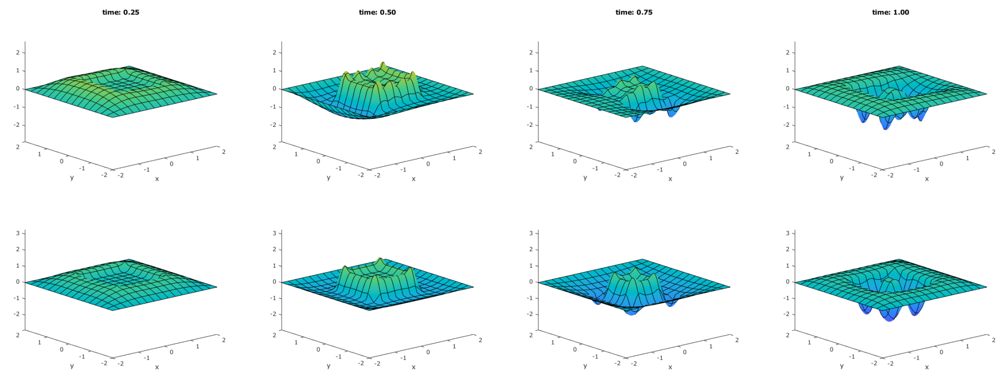


FIGURE 6. Plots of the first component of the solution of Example 5.3 (ii) u_n (top) and u^{hom} (bottom) for $t \in \{0.25, 0.5, 0.75, 1.0\}$ (left to right)

Again, we want to confirm the convergence behaviour of $U_n = (u_n, v_n)$ and its components. Our theoretical finding predicts strong convergence for u_n and weak convergence for v_n. For the numerical method, we use piecewise polynomials of degree 2 in space and one in time that are continuous in space – and therefore H^1-conforming – for the first component. For the second component, we use Raviart–Thomas-elements RT_1 of degree 1 in space and 1 in time – these are H(div)-conforming. As mesh parameter, we choose $M = 2^6$ and a tensor product mesh in space consisting of a piecewise equidistant mesh with meshsize $1/(4n)$ in $[-1, 1]$ and $1/n$ outside for the x-direction and an equidistant mesh with 80 cells in the y-direction. We fix the mesh in y-direction as the effects are anticipated to occur only in x-direction. Thus, the number of cells in total is $10 \cdot n \cdot 80$. Note that the spatial mesh resolves the stratified structure of the coefficients.

For $t \in \{0.25, 0.5, 0.75, 1.0\}$, Figure 6 shows plots of the first component of U_8 and of U^{hom}. Figure 7 investigates the convergence in the classical L_2-norm, and also weak

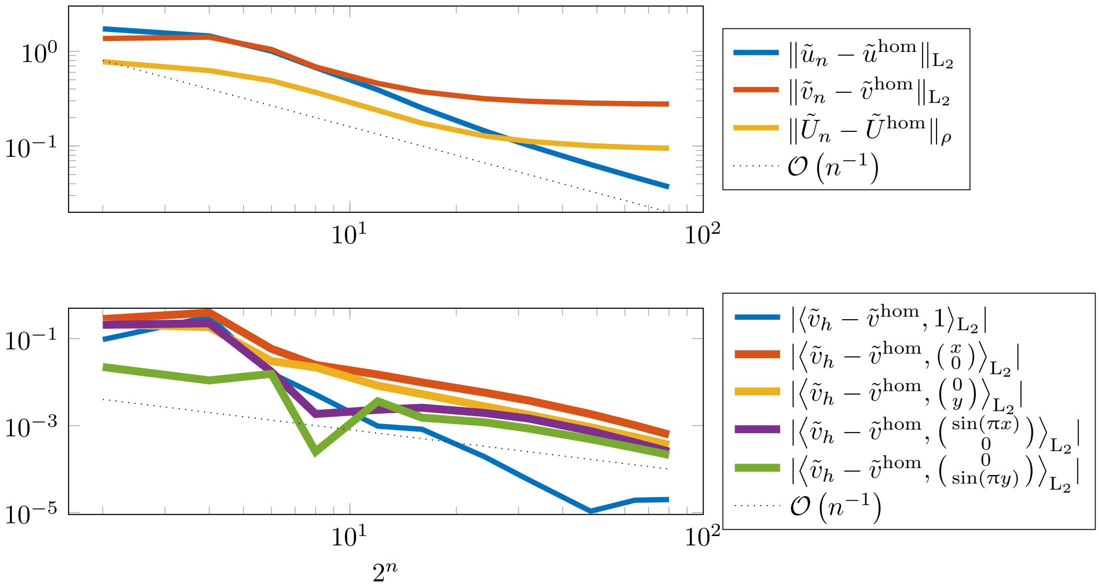


FIGURE 7. Results investigating convergence for Example 5.3 (ii)

convergence. We observe convergence of order $\mathcal{O}\left(n^{-1}\right)$ for the L_2-norm of the first component, but a stagnation in the norm-convergence of the second component and consequently of the weighted norm. Still, the second component converges weakly. Again, the graphs suggest weak convergence of order $\mathcal{O}\left(n^{-1}\right)$.

6.4. **Continuation of Example 5.3 (iii).** We once again consider the sequence of equations ($n \in \mathbb{N}$)

$$(\partial_t M_{0,n} + M_{1,n} + A)U_n = \begin{pmatrix} \sin(2\pi t) \\ 0 \end{pmatrix}$$

with a final time horizon $T = 2$. A direct approach incorporating the non-local operator in M_1 (see (19)) necessitates the inclusion of history terms as right-hand data while evaluating the time steps. This is potentially numerically cumbersome. Fortunately, we can deal with it differently:

Let us define an intrinsic variable w with $w(t, x) = 0$ for $t \leq 0$ by

$$(1 + \partial_t)^{-1} v_2 = w \quad \Leftrightarrow \quad v_2 = (1 + \partial_t)w \quad \Leftrightarrow \quad \partial_t w + w - v_2 = 0.$$

With this we rewrite the homogenised equation of Example 5.6 (iii) as

$$(\partial_t \widehat{M}_0 + \widehat{M}_1 + \widehat{A})\widehat{U}^{\mathrm{hom}} = \widehat{F}, \tag{20}$$

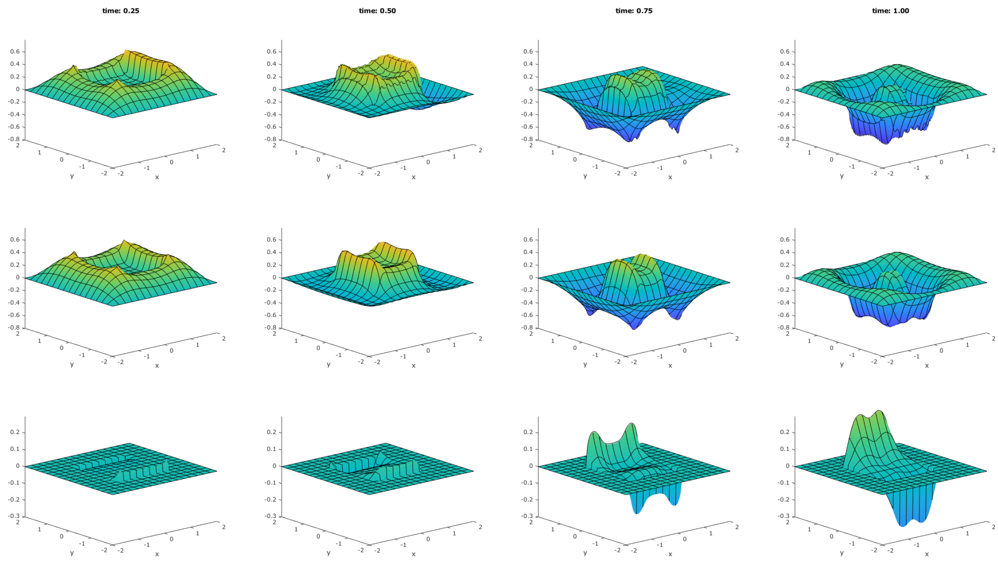


FIGURE 8. Plots of the first component of the solution U_8 of Example 5.3 (iii) (top), the first (middle) and third component (bottom) of $\widehat{U}^{\mathrm{hom}}$ of (20) for $t \in \{0.25, 0.5, 0.75, 1.0\}$ (left to right)

where $\widehat{U}^{\mathrm{hom}} := (u^{\mathrm{hom}}, v^{\mathrm{hom}}, w^{\mathrm{hom}})^{\top}$, $\widehat{F} := (f, g, 0)^{\top}$ for $F = (f, g)^{\top}$ and

$$\widehat{M_0} := \mathbb{1}_{\Omega_1} \cdot \begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & 0 & 2 \end{pmatrix} + (1 - \mathbb{1}_{\Omega_1}) \cdot \begin{pmatrix} \epsilon_0 & 0 & 0 \\ 0 & \mu_0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\widehat{M_1} := \mathbb{1}_{\Omega_1} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} & \begin{pmatrix} 0 \\ -2 \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & -2 \end{pmatrix} & 2 \end{pmatrix} + (1 - \mathbb{1}_{\Omega_1}) \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\widehat{A} := \begin{pmatrix} 0 & \operatorname{div} & 0 \\ \mathring{\operatorname{grad}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We added the equation $(1 - \mathbb{1}_{\Omega_1}) w^{\mathrm{hom}} = 0$ in order to define w^{hom} outside Ω_1. We can easily verify that (20) falls under the regime of Theorem 2.2. This formulation of the homogenised problem does not contain memory terms and the standard method developed in [FTW19] applies, if we choose an appropriate function space for the third component and set homogeneous initial conditions. As compatibility in space is not needed, discontinuous, piecewise polynomial finite elements of order one less than for the first component are possible. All in all, numerical costs are higher compared to the direct approach, but the implementation is easier.

For $t \in \{0.25, 0.5, 0.75, 1.0\}$, Figure 8 shows plots of the first component of U_8 and of U^{hom}. In addition, the pictures in the bottom row exemplify the behaviour of the intrinsic variable w^{hom}, i.e., the memory term, which apparently may not be neglected.

Figure 9 shows the convergence results of the numerical simulations. In the computation of $\tilde{U}_n$, we used a piecewise polynomials of one degree higher than in the previous calculations, because the numerical solution was still very oscillatory using quadratic elements for $\tilde{u}_n$.

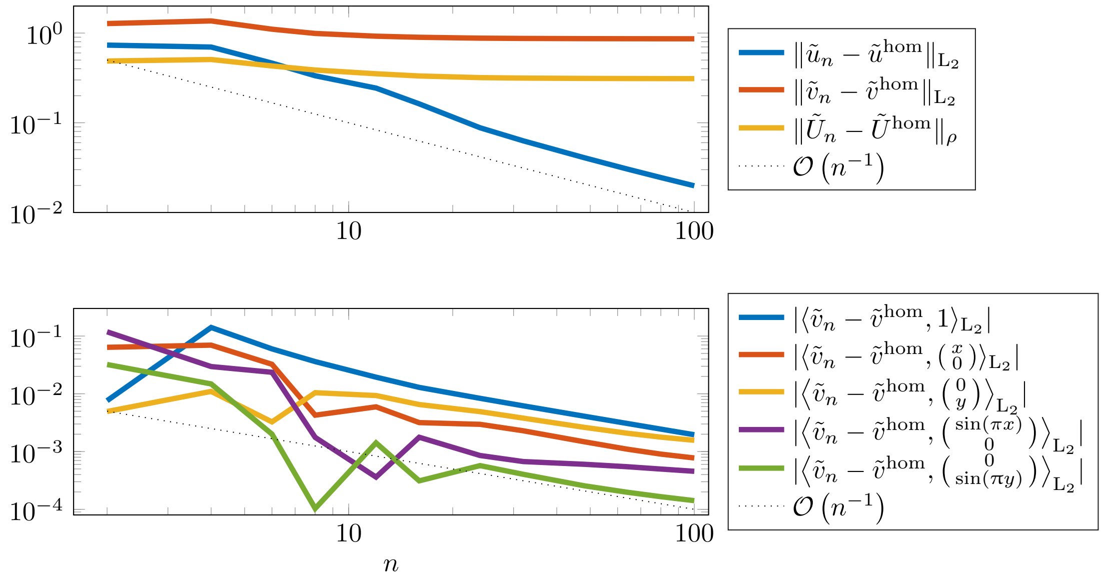


FIGURE 9. Results investigating convergence for Example 5.3 (iii) and (20)

Using higher order finite element methods can provide a natural stabilisation that reduces the oscillations, see e.g., [BR94], and we can observe first order convergence of u_n to u^{hom} in the upper graph of Figure 9. As in the previous example, v_n does not converge strongly in the L_2-norm to v^{hom}, but weakly as depicted in the lower graph.

6.5. **Continuation of Example 5.3 (iv).** We once again consider the sequence of equations ($n \in \mathbb{N}$)

$$(\partial_t M_{0,n} + M_{1,n} + A)U_n = \begin{pmatrix} \sin(2\pi t) \\ 0 \end{pmatrix}$$

with a final time horizon $T = 2$. According to Example 5.6 (iv) its homogenised limit has a memory term. In the same way as in the previous example, we can introduce an intrinsic variable $v = \sigma\varepsilon(\sigma + \varepsilon\partial_t)^{-1}E_1$ and rewrite the system in the form

$$(\partial_t \widehat{M}_0 + \widehat{M}_1 + \widehat{A})\widehat{U}^{\mathrm{hom}} = \widehat{F}, \tag{21}$$

where $\widehat{U}^{\mathrm{hom}} = (E^{\mathrm{hom}}, H^{\mathrm{hom}}, v^{\mathrm{hom}})^{\top}$, $\widehat{F} = (J, K, 0)^{\top}$ for $F = (J, K)^{\top}$ and

$$\widehat{M}_0 = \mathbb{1}_{\Omega_1} \cdot \begin{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}\epsilon & 0 \\ 0 & 0 & \frac{1}{2}\epsilon \end{pmatrix} & 0 & 0 \\ 0 & \begin{pmatrix} \frac{4}{3}\mu & 0 & 0 \\ 0 & \frac{3}{2}\mu & 0 \\ 0 & 0 & \frac{3}{2}\mu \end{pmatrix} & 0 \\ 0 & 0 & 2\epsilon \end{pmatrix} + (1 - \mathbb{1}_{\Omega_1}) \cdot \begin{pmatrix} \epsilon_0 & 0 & 0 \\ 0 & \mu_0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\widehat{M}_1 = \mathbb{1}_{\Omega_1} \cdot \begin{pmatrix} \begin{pmatrix} 2\sigma & 0 & 0 \\ 0 & \frac{1}{2}\sigma & 0 \\ 0 & 0 & \frac{1}{2}\sigma \end{pmatrix} & 0 & \begin{pmatrix} -2\sigma \\ 0 \\ 0 \end{pmatrix} \\ 0 & 0 & 0 \\ \begin{pmatrix} -2\sigma & 0 & 0 \end{pmatrix} & 0 & 2\sigma \end{pmatrix} + (1 - \mathbb{1}_{\Omega_1}) \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\widehat{A} = \begin{pmatrix} 0 & -\operatorname{curl} & 0 \\ \mathring{\operatorname{curl}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Once again, we have added the equation $(1 - \mathbb{1}_{\Omega_1})v^{\mathrm{hom}} = 0$ in order to define v^{hom} everywhere, and, once again, Theorem 2.2 renders (21) uniquely solvable. For its numerical simulation, we observe the curse of dimensions and cannot use as large numbers of n as for the previous problems. Again, we use a reference solution with a fine mesh instead of a given exact solution to the homogenised problem. Just to give an impression of its size: its number of degrees of freedom is 7 millions, 9 millions and 12 millions for the three respective components. For the stratified problem, the finest resolution had 31 millions and 40 millions degrees of freedom in its two components. The values for the differences

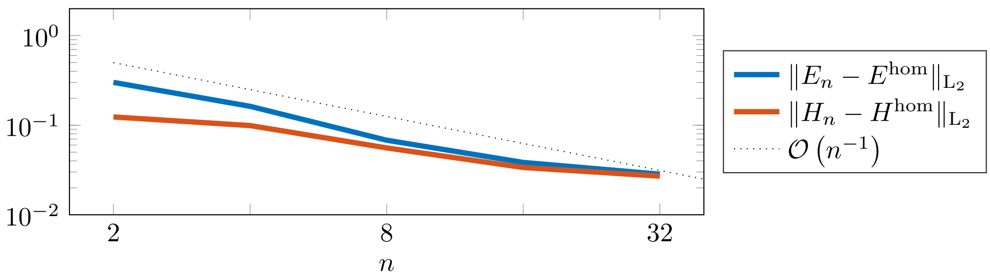


FIGURE 10. Computed errors for the full 3D Maxwell example (21).

are given in Figure 10. Both components suggest a convergence in n to the solution of the homogenised problem, however, with an order less than 1. Whether or not this behaviour can be confirmed will be further investigated looking at weak convergence. Even though we took the following small sample of test functions

$$v_1(t,\boldsymbol{x}) = \begin{pmatrix}1\\0\\0\end{pmatrix}, \qquad v_2(t,\boldsymbol{x}) = \begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad v_3(t,\boldsymbol{x}) = \begin{pmatrix}0\\0\\1\end{pmatrix},$$
$$v_4(t,\boldsymbol{x}) = \begin{pmatrix}\sin(\pi x)\\0\\0\end{pmatrix}, \qquad v_5(t,\boldsymbol{x}) = \begin{pmatrix}0\\\sin(\pi y)\\0\end{pmatrix}, \qquad v_6(t,\boldsymbol{x}) = \begin{pmatrix}0\\0\\\sin(\pi z)\end{pmatrix},$$
$$v_7(t,\boldsymbol{x}) = \begin{pmatrix}xy\\0\\0\end{pmatrix}, \qquad v_8(t,\boldsymbol{x}) = \begin{pmatrix}0\\y^2+z\\0\end{pmatrix}, \qquad v_9(t,\boldsymbol{x}) = \begin{pmatrix}0\\0\\xyz\end{pmatrix},$$

a similar convergence behaviour cannot be confirmed as Figure 11 shows (the results

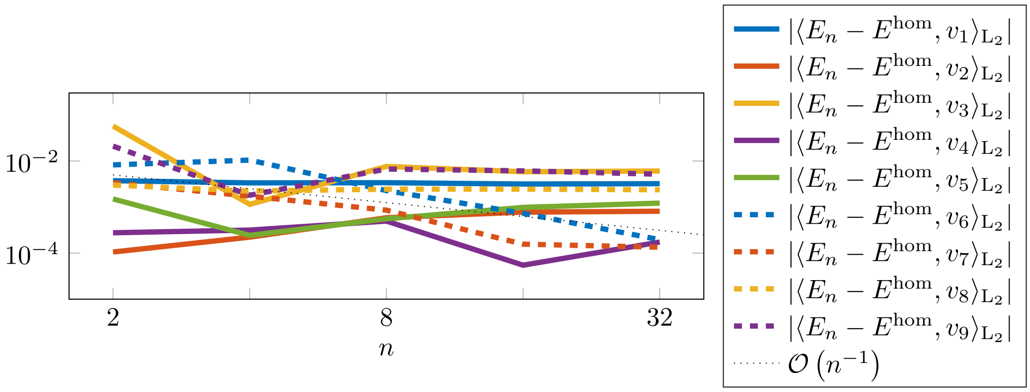


FIGURE 11. Investigation of probable weak convergence of the full 3D Maxwell example (21).

for H_n are similar). We are probably still far away from the convergent regime, but at the same time at the end of our computational possibilities. Although our theoretical findings assert that the numbers will go down eventually, this did not happen numerically for $n \leq 32$.

7. Conclusion

We have applied an abstract continuous dependence result to homogenisation problems for ordinary differential equations as well as partial differential equations of potentially mixed type. Further theoretical insight led to norm-convergence statements, where only weak convergence was explicitly known. Numerical findings support these abstract results.

Acknowledgements

M.W. acknowledges useful discussions with Dirk Pauly and Rainer Picard concerning the Helmholtz decomposition in 2D.

Data Availability

The data leading to the present study can be shared upon personal request to S.F.

Appendix A.

Lemma A.1. *For $d \in \mathbb{N}$, an open $\Omega \subseteq \mathbb{R}^d$, and a skew-selfadjoint $C \in \mathbb{R}^{d\times d}$,*

$$\langle C \mathring{\operatorname{grad}}\, u, \mathring{\operatorname{grad}}\, v\rangle_{L_2(\Omega)^d} = 0$$

holds for all $u, v \in \mathring{H}^1(\Omega)$.

Proof. Clearly, we only have to prove the claim for $u, v \in C_c^\infty(\Omega)$. In that case integration by parts yields ($C = (c_{ij})_{1\le j,k\le d}$)

$$\langle C \mathring{\operatorname{grad}}\, u, \mathring{\operatorname{grad}}\, v\rangle_{L_2(\Omega)^d} = \int_\Omega \sum_{j=1}^d \sum_{k=1}^d c_{jk}\overline{\partial_k u}\partial_j v \,\mathrm{d}x = -\int_\Omega \overline{u} \sum_{j=1}^d \sum_{k=1}^d c_{jk}\partial_k \partial_j v \,\mathrm{d}x \tag{22}$$

The skew-selfadjointness and Schwarz's theorem imply

$$c_{jk}\partial_k\partial_j v = -c_{kj}\partial_k\partial_j v = -c_{kj}\partial_j\partial_k v$$

for $1 \le j, k \le d$. With that, we can immediately deduce that (22) vanishes. ❑

Lemma A.2. *Let $a \in \mathbb{R}^{d\times d}$. Then*

$$\Big(\forall x \in \mathbb{R}^d\colon \langle ax, x\rangle_{\mathbb{R}^d} \ge \alpha\|x\|^2_{\mathbb{R}^d}\Big) \qquad \Leftrightarrow \qquad \Big(\forall z \in \mathbb{C}^d\colon \operatorname{Re}\langle az, z\rangle_{\mathbb{C}^d} \ge \alpha\|z\|^2_{\mathbb{C}^d}\Big)$$

Proof. Clearly the implication "$\Leftarrow$" is true. Hence, let $z = x + \mathrm{i}y \in \mathbb{C}^d$, where $x, y \in \mathbb{R}^d$. Then

$$\begin{aligned}\operatorname{Re}\langle az, z\rangle_{\mathbb{C}^d} &= \operatorname{Re}\big(\langle ax, x\rangle_{\mathbb{C}^d} + \langle ax, \mathrm{i}y\rangle_{\mathbb{C}^d} + \langle a\mathrm{i}y, x\rangle_{\mathbb{C}^d} + \langle a\mathrm{i}y, \mathrm{i}y\rangle_{\mathbb{C}^d}\big) = \langle ax, x\rangle_{\mathbb{R}^d} + \langle ay, y\rangle_{\mathbb{R}^d} \\ &\ge \alpha(\|x\|^2_{\mathbb{R}^d} + \|y\|^2_{\mathbb{R}^d}) = \alpha\|z\|^2_{\mathbb{C}^d}.\end{aligned}$$ ❑

Theorem A.3. *For $d \in \mathbb{N}$, consider $f \in L_\infty(\mathbb{R}^d)$ with $f(\cdot + k) = f$ for all $k \in \mathbb{Z}^d$. Furthermore, let $\Omega \subseteq \mathbb{R}^d$ be measurable with non-zero measure. Then, the sequence of bounded multiplication operators $(f(n\cdot))_{n\in\mathbb{N}}$ in $\mathcal{L}_b(L_2(\Omega))$ converges to $\int_{(0,1)^d} f(x)\,\mathrm{d}x$ in the weak operator topology.*

Proof. See, e.g., [STW22, Theorem 13.2.4]. ❑

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