Mixed-Dimensional Geometric Coupling of Port-Hamiltonian Systems

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Abstract

We propose a new interconnection relation for infinite-dimensional port-Hamiltonian systems that enables the coupling of ports with different spatial dimensions by integrating over the the surplus dimensions. To show the practical relevance, we apply this interconnection to a model system of an actively cooled gas turbine blade. We also show that this interconnection relation behaves well with respect to a discretization in finite element space, ensuring its usability for practical applications.

Keywords: Port-Hamiltonian System, Coupled Systems, Geometric Coupling, Infinite Dimensional Systems
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1. Motivation

Scientific models are inherently approximations of reality, and removing unnecessary details can greatly simplify the resulting model. These simplifications often involve reducing the spatial dimensions of the model: A fluid flowing through a pipe is often modelled in 1D rather than using the full 3D Navier-Stokes equations. Electronic components such as capacitors and resistors are commonly modelled as 0D elements. When the interfaces of the subsystems have the same dimension, there are formalisms such as *Port-Hamiltonian Systems* (PHS) that treat the interconnection of these systems in a fairly general way. For more details on the background of port-Hamiltonian systems we refer the reader to [1–3, 10].

However, it becomes difficult when the subsystems have different spatial dimensions. For example, modeling a one-dimensional pipe flow that interacts with its environment via the pipe walls requires coupling a 1D interface (the fluid flow) with a 2D interface (the pipe walls). Coupling the pipe walls to a

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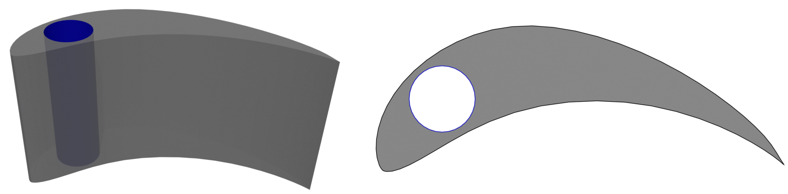

(a) 3D view (b) Top view

Figure 1: Simple model of a cooled turbine blade, with the cooling channel in blue.

lumped-parameter model for the temperature of the room in which they are located requires coupling the 2D pipe surface to a zero-dimensional system.

In the following sections, we will attempt to formulate an energy-conserving connection of two port-Hamiltonian systems where the connected ports do not have the same spatial dimension.

2. Motivating Example: Cooled Gas Turbine Blade

Consider the heat flow in a gas turbine blade cooled by an internal cooling channel, as shown in Figure 1. We can model this as two interconnected subsystems: the heat conduction within the metal of the turbine blade and the coolant flow within the cooling channel. In order to couple these systems, we introduce the following decomposition of the boundary $\partial\Omega$ of the turbine plate Ω. As displayed in Figure 1 the boundary can be split into Γ_{int} the boundary to the cooling channel and exterior boundary Γ_{ext}, i.e. $\partial\Omega = \Gamma_{\text{int}} \dot{\cup} \Gamma_{\text{ext}}$. For more information and a discussion of a greatly simplified version of this system, see [4].

Heat conduction in the metal is, of course, modelled by a heat equation:

$$\rho c \frac{\partial T_h}{\partial t}(x,t) = \operatorname{div}\big(\lambda \operatorname{grad} T_h(x,t)\big). \tag{1}$$

The formulation as a port-Hamiltonian system closely follows [8], choosing the thermal energy U as Hamiltonian

$$U(t) = \int_\Omega q\big(s(x,t)\big)\,\mathrm{d}x, \tag{2}$$

and considering the thermal energy density q as a function of the entropy density s such that the thermodynamic relation $\delta_s U = \frac{\mathrm{d}q}{\mathrm{d}s} = T_h$ is satisfied. Taking s as a state variable, we obtain the usual flow $f_s = \frac{\partial s}{\partial t}$ and the corresponding effort $e_s = T_h$ (the power conjugated quantities). As additional flows and efforts we choose the entropy flux $e_\Phi = \Phi_S$, as well as $f_\Phi = -\operatorname{grad}(T_h)$, $f_\sigma = T_h$ and $e_\sigma = -\operatorname{grad}(\frac{1}{T_h})\Phi_U$ with the heat flux Φ_U. Thus, we obtain the port-Hamiltonian system

$$\begin{pmatrix} f_s \\ f_\Phi \\ f_\sigma \end{pmatrix} = \begin{pmatrix} 0 & -\operatorname{div} & -1 \\ -\operatorname{grad} & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_s \\ e_\Phi \\ e_\sigma \end{pmatrix}. \tag{3}$$

Since (3) has one algebraic equation, we add the two *closure relations*

$$e_s e_\Phi = \lambda f_\Phi \quad \text{and} \quad f_\Phi e_\Phi = -f_\sigma e_\sigma, \tag{4}$$

the former being Fourier's law and the latter expressing the relation between heat flux Φ_U and entropy flux Φ_S. The system (3)–(4) is complemented with the following *boundary conditions* modelling the energy flow across the boundary

$$\text{input:} \quad u_1 = T_h|_{\Gamma_{\text{int}}}, \qquad u_2 = T_h|_{\Gamma_{\text{ext}}} \tag{5}$$

$$\text{and} \quad \text{output:} \quad v_1 = -(\Phi_S \cdot \vec{n})|_{\Gamma_{\text{int}}}, \quad v_2 = -(\Phi_S \cdot \vec{n})|_{\Gamma_{\text{ext}}} \tag{6}$$

with $\vec{n}$ being the surface normal vector.

The coolant flow in the cooling channel is modelled as a 1D compressible fluid. This is consistent with common practice in engineering, since cooling channels in practice are small, irregularly shaped, and exhibit highly turbulent flow, making full 3D flow models infeasible for practical applications and requiring the use of 1D parameter models, such as those presented in [6]. A 1D model also allows us to use the formulation of irreversible PHS with boundary control presented in [7]. We choose the specific volume $\varphi = 1/\rho$, the velocity v and the entropy density s as state variables, and the Hamiltonian

$$H(v, \varphi, s) = \int_a^b \Big(\frac{1}{2} v^2 + u(\varphi, s) \Big) \, \mathrm{d}z, \tag{7}$$

where the internal energy density u fulfils the Gibbs relation $\mathrm{d}u = -p \, \mathrm{d}\varphi + T_c \, \mathrm{d}s$. We can then formulate the quasi-Hamiltonian system

$$\begin{pmatrix} \frac{\partial \varphi}{\partial t} \\ \frac{\partial v}{\partial t} \\ \frac{\partial s}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\partial}{\partial z} & 0 \\ \frac{\partial}{\partial z} & 0 & -\frac{fv}{T} \\ 0 & \frac{fv}{T} & 0 \end{pmatrix} \begin{pmatrix} -p \\ v \\ T_c \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} w_1(z, t), \tag{8}$$

$$y_1 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -p \\ v \\ T_c \end{pmatrix} = T_c, \tag{9}$$

with the appropriate boundary ports w_2, y_2 for inflow and outflow of the cooling channel. This system is an infinite-dimensional irreversible port-Hamiltonian system as defined in [7, Definition 1].

Coupling the two systems using the usual interconnections for PHS, like Robin-type heat flow conditions, does not work because the spatial dimensions do not match: The boundary port of the heat equation is 2D, while the distributed port of the cooling channel is only 1D. We need a new interconnection to compensate for this dimensional mismatch. We will do this by introducing an additional operator A and its adjoint. This is displayed in Figure 2. The coupling ports can be seen as an own port-Hamiltonian system or equivalently as a Dirac structure, as we will show in the next section. However, going into the functional analytic details of this interconnection is beyond the scope of this work. The difficulty is to handle the fractional Sobolev spaces on the boundary Γ_{int} that appear as input and output spaces of the 3D heat equation.

Note that we are aiming to interconnect boundary ports (from the 3D model) with distributed ports (from the 1D model)

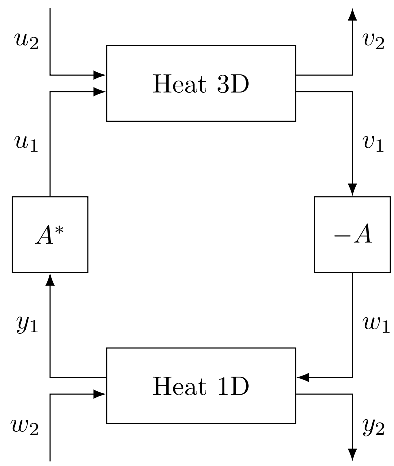

Figure 2: Diagram of the coupling

3. Proposition: Mixed-Dimensional Geometric Coupling

Definition 3.1 (Dirac structure [5])**.** Let $\mathcal{F}$ be a linear space, $\mathcal{E}$ its dual and $\langle \cdot, \cdot \rangle \colon \mathcal{E} \times \mathcal{F} \to \mathbb{R}$ their dual product. Further let

$$\left\langle\!\!\left\langle \begin{pmatrix} e_1 \\ f_1 \end{pmatrix}, \begin{pmatrix} e_2 \\ f_2 \end{pmatrix} \right\rangle\!\!\right\rangle = \langle e_1, f_2 \rangle + \langle e_2, f_1 \rangle \qquad \begin{pmatrix} e_1 \\ f_1 \end{pmatrix}, \begin{pmatrix} e_2 \\ f_2 \end{pmatrix} \in \mathcal{E} \times \mathcal{F}. \tag{10}$$

Then $\mathcal{D} \subseteq (\mathcal{E} \times \mathcal{F})$ is a *Dirac structure* if $\mathcal{D} = \mathcal{D}^\perp$ with

$$\mathcal{D}^\perp = \{a \in \mathcal{E} \times \mathcal{F} \mid \langle\!\langle a, b \rangle\!\rangle = 0 \quad \forall\, b \in \mathcal{D}\}. \tag{11}$$

Let us remark that $\mathcal{E}$ and $\mathcal{F}$ here not only contain the storage ports, but can also contain dissipative ports and external ports – which, in the case of Stokes-Dirac structures contain distributed ports and boundary ports. The most relevant difference between a Dirac structure and a Stokes-Dirac structure is the presence of a boundary port, which regulates the flow of energy across the boundary and takes the place of the boundary conditions in "normal" PDEs.

Theorem 3.2. *Let $\Gamma_1 \subseteq \mathbb{R}^n$ compact, $\Gamma_2 \subset \mathbb{R}^m$ compact and $\Gamma \coloneqq \Gamma_1 \times \Gamma_2 \subseteq \mathbb{R}^{n+m}$. Further let $\mathcal{F} = \mathrm{L}^2(\Gamma_1) \times \mathrm{L}^2(\Gamma)$ and $\mathcal{E} = \mathcal{F}^*$ its dual. Note that we have for $x \in \Gamma$ the decomposition $x = (x_1, x_2)$ with $x_1 \in \Gamma_1$ and $x_2 \in \Gamma_2$. Finally, let*

$$A\colon \begin{cases} \mathrm{L}^2(\Gamma) & \to \quad \mathrm{L}^2(\Gamma_1), \\ \quad u & \mapsto \quad \int_{\Gamma_2} u(\cdot, x_2)\, \mathrm{d}x_2, \end{cases} \tag{12}$$

and the embedding

$$B\colon \begin{cases} \mathrm{L}^2(\Gamma_1) & \to \quad \mathrm{L}^2(\Gamma), \\ \quad v & \mapsto \quad v. \end{cases} \tag{13}$$

The previous operator has to be understood as $(Bv)(x_1, x_2) = v(x_1)$. Then

$$J\colon \begin{cases} \mathcal{E} & \to \quad \mathcal{F}, \\ e & \mapsto \quad \begin{pmatrix} 0 & -A \\ B & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \end{cases} \tag{14}$$

induces a Dirac structure

$$\mathcal{D} = \big\{(e, f) \in \mathcal{E} \times \mathcal{F} \mid f = Je\big\}. \tag{15}$$

Note that $u \in \mathrm{L}^2(\Gamma)$ implies $u(\cdot, x_2) \in \mathrm{L}^2(\Gamma_1)$ for almost every $x_2 \in \Gamma_2$. Moreover, by the triangle inequality and Cauchy-Schwarz inequality

$$\begin{aligned} \|Au\|^2_{\mathrm{L}^2(\Gamma_1)} &= \int_{\Gamma_1} \left| \int_{\Gamma_2} u(x_1, x_2)\, \mathrm{d}x_2 \right|^2 \mathrm{d}x_1 \le \int_{\Gamma_1} \Big(\int_{\Gamma_2} 1 \cdot |u(x_1, x_2)|\, \mathrm{d}x_2 \Big)^2 \mathrm{d}x_1 \\ &\overset{C.S.}{\le} |\Gamma_2| \int_{\Gamma_1} \int_{\Gamma_2} |u(x_1, x_2)|^2\, \mathrm{d}x_2\, \mathrm{d}x_1 = |\Gamma_2| \|u\|^2_{\mathrm{L}^2(\Gamma)}, \end{aligned} \tag{16}$$

where $|\Gamma_2|$ denotes the measure of Γ_2. Hence, the operator A is well-defined. Note that this holds true for any finite measure on Γ_2. In particular we will later use surface measures.

Proof. Determine the adjoint operator of B: For $f \in \mathrm{L}^2(\Gamma)$, $v \in \mathrm{L}^2(\Gamma_1)$ we have

$$\begin{aligned} \langle f, Bv \rangle_{L^2(\Gamma)} &= \int_{\Gamma_1} \int_{\Gamma_2} f(x_1, x_2) v(x_1)\, \mathrm{d}x_2\, \mathrm{d}x_1 = \int_{\Gamma_1} \Big(\int_{\Gamma_2} f(x_1, x_2)\, \mathrm{d}x_2 \Big) v(x_1)\, \mathrm{d}x_1 \\ &= \Big\langle \int_{\Gamma_2} f(\cdot, x_2)\, \mathrm{d}x_2, v \Big\rangle_{\mathrm{L}^2(\Gamma_1)} = \langle B^* f, v \rangle_{\mathrm{L}^2(\Gamma_1)} = \langle Af, v \rangle_{\mathrm{L}^2(\Gamma_1)}. \end{aligned} \tag{17}$$

Since $A = B^*$ holds, J is skew-adjoint and $\mathcal{D}$ is a Dirac structure [9]. □

4. Coupled Example System

To apply the coupling described in Section 3 to the system of Section 2, we recall the splitting of the boundary $\partial\Omega$ of the 3D heat equation domain Ω into an external part Γ_{ext}, which connects to the outside of the blade and is disregarded

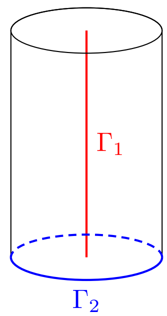

Figure 3: Illustration of $\Gamma_{\text{int}} \cong \Gamma_1 \times \Gamma_2$

here, and an internal part Γ_{int} which denotes the wall of the cooling channel and will be coupled to the coolant flow.

As the cooling channel is modelled as a tube, it can be decomposed into $\Gamma_{\text{int}} \cong \Gamma_1 \times \Gamma_2$ as in Theorem 3.2, with Γ_1 containing the axial coordinate (along the flow direction) and Γ_2 the azimuthal coordinate, i.e. describing the circumference, see Figure 3. The temperature T_h and T_c, an intensive quantity, of the points that are in contact with each other is the same, while the entropy flux Φ_S, an extensive quantity, is integrated and has the expected sign change. Based on this physical considerations we choose the following interconnection. Note that y_1 is the output that corresponds to the cooling channel that is modelled by the 1-D system. Hence, we can say the domain of y_1 is Γ_1

$$u_1 = T_h\big|_{\Gamma_{\text{int}}} = BT_c = By_1 \quad \text{and} \quad w_1 = \int_{\Gamma_2} \Phi_S(x) \cdot \vec{n} \, \mathrm{d}x_2 = -\underbrace{\int_{\Gamma_2} v_1 \, \mathrm{d}x_2}_{=Av_1}, \tag{18}$$

This interconnection has exactly the form given in Theorem 3.2. Since it is an energy preserving interconnection, the coupled system is a (quasi-)Hamiltonian system and would be a port-Hamiltonian system if both sub-systems were PHS.

5. Finite Element Discretization

The interconnection proposed in Section 3 can be easily discretized with a finite element discretization. The result will then be a finite-dimensional Dirac structure, as we will see in this section.

Let us assume that we have finite element discretizations for both sub-systems, with ψ_i the basis functions on the boundary of the higher-dimensional system (the heat equation in our example), and χ_i the basis functions of the lower-dimensional system (the compressible cooling fluid in our example). We can then approximate the input u and output v of the first system, and the input w

and output y of the second system as

$$\begin{aligned} u &\approx \sum_i \psi_i(x)u_i(t) = \Psi^\top(x)\underline{u}(t), & v &\approx \sum_i \psi_i(x)v_i(t) = \Psi^\top(x)\underline{v}(t), \\ w &\approx \sum_i \chi_i(x_1)w_i(t) = X^\top(x_1)\underline{w}(t), & y &\approx \sum_i \chi_i(x_1)y_i(t) = X^\top(x_1)\underline{y}(t). \end{aligned} \tag{19}$$

Remembering that $x = (x_1, x_2)^\top$ and applying these approximations to the continuous interconnection relations of Equation (18) results in

$$X^\top(x_1)\underline{w}(t) = -\int_{\Omega_2} \Psi^\top(x)\underline{v}(t)\,\mathrm{d}x_2, \quad \text{and} \quad \Psi^\top(x)\underline{u}(t) = X^\top(x_1)\underline{y}(t). \tag{20}$$

We now take the weak form of Equation (20) to obtain the discretized forms of the interconnection relations

$$\begin{aligned} M_\chi\underline{w}(t) &= \int_{\Gamma_1} X(x_1)X^\top(x_1)\underline{w}(t)\,\mathrm{d}x_1 = -\int_{\Gamma_1} X(x_1)\int_{\Gamma_2} \Psi^\top(x)\underline{v}(t)\,\mathrm{d}x_2\,\mathrm{d}x_1 \\ &= -\int_{\Gamma_1} X(x_1)\widehat{\Psi}^\top(x_1)\underline{v}(t)\,\mathrm{d}x_1 = -D_\chi\underline{v}(t) \end{aligned} \tag{21}$$

and

$$\begin{aligned} M_\psi\underline{u}(t) &= \int_\Gamma \Psi(x)\Psi^\top(x)\underline{u}(t)\,\mathrm{d}x = \int_\Gamma \Psi(x)X^\top(x_1)\underline{y}(t)\,\mathrm{d}x \\ &= \int_{\Gamma_1}\left(\int_{\Gamma_2}\Psi(x)\,\mathrm{d}x_2\right)X^\top(x_1)\underline{y}(t)\,\mathrm{d}x_1 \\ &= \int_{\Gamma_1}\widehat{\Psi}(x_1)X^\top(x_1)\underline{y}(t)\,\mathrm{d}x_1 = D_\psi\underline{y}(t). \end{aligned} \tag{22}$$

Since $D_\psi = D_\chi^\top$, the discretized interconnection relation

$$\begin{pmatrix} M_\chi & 0 \\ 0 & M_\psi \end{pmatrix}\begin{pmatrix} \underline{u}(t) \\ \underline{w}(t) \end{pmatrix} = \begin{pmatrix} 0 & -D_\chi \\ D_\psi & 0 \end{pmatrix}\begin{pmatrix} \underline{v}(t) \\ \underline{y}(t) \end{pmatrix} \tag{23}$$

represents a Dirac structure.

Remark 1. The integration over Γ_2 will not expand the support of the basis functions $\widehat{\psi}_i$ in x_1-direction. Therefore, the matrix D_χ will still be sparse, although less sparse than the matrix M_χ.

6. Conclusion

It is possible to couple port-Hamiltonian systems of different spatial dimensions if the interconnecting ports do not have the same spatial dimension. The proposed interconnection structure forms a Dirac structure and thus ensures that the resulting overall system again forms a port-Hamiltonian system.

Application to an example system has shown that the interconnection has practical use and a physically meaningful interpretation when the ports consist of both extensive and intensive variables. This is usually the case for physically motivated port-Hamiltonian systems, but cannot be guaranteed in general.

Finally, we showed that the interconnection behaves well with respect to the discretization in finite element space, leading to a finite-dimensional Dirac structure.

References

[1] C. Beattie, V. Mehrmann, H. Xu, and H. Zwart, *Linear Port-Hamiltonian Descriptor Systems*, Math. Control Signals Syst. 30(4) (2018), 1–27.

[2] V. Duindam, A. Macchelli, S. Stramigioli, and H. Bruyninckx (eds.), *Modeling and Control of Complex Physical Systems – The Port-Hamiltonian Approach*, Springer 2009.

[3] B. Jacob and H. Zwart, *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*, Springer 2012.

[4] J. Jäschke, M. Ehrhardt, M. Günther, and B. Jacob, *A port-Hamiltonian formulation of coupled heat transfer*, Math. Comput. Model. Dyn. Syst. 28(1) (2022), 78–94.

[5] Y. Le Gorrec, H. Zwart, and B. Maschke, *Dirac structures and boundary control systems associated with skew-symmetric differential operators*, SIAM J. Contr. Optim. 44(5) (2005), 1864–1892,

[6] P.L. Meitner, *Computer code for predicting coolant flow and heat transfer in turbomachinery*, Technical report, NASA, 1990.

[7] H. Ramirez, Y. Le Gorrec, and B. Maschke, *Boundary controlled irreversible port-Hamiltonian systems*, Chem. Engrg. Sci. 248 (2022), 117107.

[8] A. Serhani, G. Haine, and D. Matignon, *Anisotropic heterogeneous n-D heat equation with boundary control and observation: I. Modeling as port-Hamiltonian system*, IFAC-PapersOnLine 52(7) (2019), 51–56.

[9] A.J. van der Schaft and B.M. Maschke, *Hamiltonian formulation of distributed-parameter systems with boundary energy flow*, J. Geom. Phys. 42(1) (2002), 166–194.

[10] A. van der Schaft, *Port-Hamiltonian systems: an Introductory Survey*, in: M. Sanz-Sole, J. Soria, J.L. Varona, and J. Verdera (eds.), Proc. Int. Congress of Mathematicians, Vol. III, Madrid, Spain, 2006, pp. 1339–1365.