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WELL-POSEDNESS OF LINEAR FIRST ORDER PORT-HAMILTONIAN SYSTEMS ON MULTIDIMENSIONAL SPATIAL DOMAINS

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Abstract. We consider a port-Hamiltonian system on an open spatial domain $\Omega \subseteq \mathbb{R}^n$ with bounded Lipschitz boundary. We show that there is a boundary triple associated to this system. Hence, we can characterize all boundary conditions that provide unique solutions that are non-increasing in the Hamiltonian. As a by-product we develop the theory of quasi Gelfand triples. Adding "natural" boundary controls and boundary observations yields scattering/impedance passive boundary control systems. This framework will be applied to the wave equation, Maxwell's equations and Mindlin plate model. Probably, there are even more applications.

1. **Introduction.** The aim of this paper is to develop a port-Hamiltonian framework on multidimensional spatial domains that justifies existence and uniqueness of solutions. Those systems can be described by the following equations

$$
\begin{aligned}
\frac{\partial}{\partial t} x(t,\zeta) &= \sum_{i=1}^{n} \frac{\partial}{\partial \zeta_i} P_i\big(\mathcal{H}(\zeta)x(t,\zeta)\big) + P_0\big(\mathcal{H}(\zeta)x(t,\zeta)\big), && \zeta \in \Omega, t \geq 0,\\
x(0,\zeta) &= x_0(\zeta), && \zeta \in \Omega,
\end{aligned}
$$

where x is the state, P_i and P_0 are matrices, $\mathcal{H}$ is the Hamiltonian density, and Ω is an open subset of $\mathbb{R}^n$ with bounded Lipschitz boundary. We will restrict ourselves to the case, where the matrices P_i have the block shape $\begin{bmatrix} 0 & L_i \\ L_i^{\mathsf{H}} & 0 \end{bmatrix}$ for $i \in \{1,\dots,n\}$. We also introduce "natural" boundary controls and observations which make the system a scattering passive (energy preserving) or impedance passive (energy preserving) boundary control system. This PDE perfectly matches the description of port-Hamiltonian systems in one spatial dimension in [8], if we set $n = 1$. The additional restriction $P_1 = \begin{bmatrix} 0 & L_1 \\ L_1^{\mathsf{H}} & 0 \end{bmatrix}$ is not needed in [8], since the boundary of a line automatically satisfies certain symmetry properties. We decided to not demand an analogous symmetry from Ω in the multidimensional case, because

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it did not seem very restrictive to ask for $P_i = \begin{bmatrix} 0 & L_i \\ L_i^{\mathsf{H}} & 0 \end{bmatrix}$ as all the examples satisfy this anyway. However, it is probably possible to drop this restriction and ask instead for certain a symmetry of the boundary.

The port-Hamiltonian formulation has proven to be a powerful tool for the modeling and control of complex multiphysics systems. An introductory overview can be found in [14]. For one-dimensional spatial domains concerns about existence and uniqueness of solutions are covered in [8].

Chapter 8 of the Ph.D. thesis [15] also regards such port-Hamiltonian systems that have multidimensional spatial domains, but the results demand very strong assumptions on the boundary operators (they have to map into $H^{1/2}(\partial\Omega)^k$ and its dual respectively), which are in case of Maxwell's equations and the Mindlin plate model not satisfied, as Example A.4 shows for Maxwell's equations. With the following approach we will overcome these limits.

The strategy is to find a boundary triple associated to the differential operator. The multidimensional integration by parts formula already suggests possible operators for a boundary triple, but unfortunately these operators cannot be extended to the entire domain of the differential operator. Hence, we need to adapt the codomain of these boundary operators, which will lead to the construction of suitable boundary spaces for this problem. These boundary spaces behave like a Gelfand triple with the original codomain as pivot space, but lack of a chain inclusion.

Up to the author's best knowledge there is no earlier theory about this setting. So we will develop the notion of *quasi Gelfand triples* in section 5, which equips us with the tools to state the boundary condition in terms of the pivot space instead of the artificially constructed boundary spaces (Theorem 7.6). Section 5 can be read isolated from the rest.

One can think of using a quasi boundary triple $(\mathcal{G}, \Gamma_0, \Gamma_1)$ (see [1]) to overcome the extension problem of the boundary mappings, but unfortunately the condition $\ker \Gamma_0$ is self-adjoint (or in this setting skew-adjoint) is in general not satisfied for our purpose.

The approach to the wave equation in [9] perfectly fits the framework presented in this paper. In fact, many ideas from [9] are generalized in this work. Also Maxwell's equations can be formulated as such a port-Hamiltonian system and the results in [16] can also be derived with the tools of this paper. Moreover, this theory can be applied on the model of the Mindlin plate in [2, 10]. In section 8 we give examples of how this framework can be applied to these three PDEs.

Symbols.

Symbol	Meaning	Page
$B_r(\zeta_0)$	$\{\zeta \in X : \lVert\zeta - \zeta_0\rVert_X < r\}$ ball with radius r and center ζ_0 in a normed space X	
$\mathcal{D}(\Omega)$	set of $C^\infty(\Omega)$ functions with compact support	969
$\mathcal{D}'(\Omega)$	(anti)dual space of $\mathcal{D}(\Omega)$	969
$\mathcal{D}(\mathbb{R}^n)\big\vert_\Omega$	$\{f\big\vert_\Omega : f \in \mathcal{D}(\mathbb{R}^n)\}$	969
ν	outward pointing normed normal vector on $\partial\Omega$	969
γ_0	$H^1(\Omega, X) \to L^2(\partial\Omega, X)$; extension of $f \mapsto f\big\vert_{\partial\Omega}$	969
L_∂	$\sum_{i=1}^n \partial_i L_i$ a differential operator from $L^2(\Omega)^{m_2}$ to $L^2(\Omega)^{m_1}$	969

L_∂^{H}	$\sum_{i=1}^n \partial_i L_i^{\mathsf{H}}$	969
$H(L_\partial, \Omega)$	$\{f \in L^2(\Omega)^{m_2} : L_\partial f \in L^2(\Omega)^{m_1}\}$ maximal domain of L_∂	969
$H_0(L_\partial, \Omega)$	The closure of $\mathcal{D}(\Omega)^{m_2}$ in $H(L_\partial, \Omega)$	969
$H_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega)$	$\ker \bar{\pi}_L^{\Gamma_0}$	989
L_ν	$\sum_{i=1}^n \nu_i L_i \colon L^2(\partial\Omega)^{m_2} \to L^2(\partial\Omega)^{m_1}$	969
$\bar{L}_\nu^\Gamma$	$H(L_\partial, \Omega) \to \mathcal{V}_{L,\Gamma}'$; extension of $\mathbb{1}_\Gamma L_\nu \gamma_0$ on $H(L_\partial, \Omega)$	989
$\bar{L}_\nu$	$\bar{L}_\nu^{\partial\Omega} \colon H(L_\partial, \Omega) \to \mathcal{V}_L'$; extension of $L_\nu \gamma_0$ on $H(L_\partial, \Omega)$	990
$L_\pi^2(\Gamma)$	$\overline{\operatorname{ran} \mathbb{1}_\Gamma L_\nu \gamma_0} \subseteq L^2(\Gamma)^{m_1}$	987
π_L^Γ	$H^1(\Omega)^{m_1} \to L_\pi^2(\Gamma)$; projection on $L_\pi^2(\Gamma)$ composed with γ_0	987
π_L	$\pi_L^{\partial\Omega} \colon H^1(\Omega)^{m_1} \to L^2(\partial\Omega)^{m_1}$	987
$\bar{\pi}_L^\Gamma$	$H(L_\partial^{\mathsf{H}}, \Omega) \to \mathcal{V}_{L,\Gamma}$; extension of π_L^Γ on $H(L_\partial^{\mathsf{H}}, \Omega)$	988
$\bar{\pi}_L$	$\bar{\pi}_L^{\partial\Omega} \colon H(L_\partial^{\mathsf{H}}, \Omega) \to \mathcal{V}_L$	988
M_Γ	$\operatorname{ran} \pi_L^\Gamma \subseteq L_\pi^2(\Gamma)$	988
$\mathcal{V}_{L,\Gamma_1}$	$\operatorname{ran} \bar{\pi}_L\big\vert_{H_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega)}$	989
$\mathcal{V}_L$	$\mathcal{V}_{L,\partial\Omega}$	989
$\mathcal{H}$	Hamiltonian density	976
$\mathcal{X}_\mathcal{H}$	$L^2(\Omega)^m$ equipped with $\langle \mathcal{H}., .\rangle_{L^2(\Omega)^m}$; the state space	976
$\begin{bmatrix} (\mathcal{H}x)_{L^{\mathsf{H}}} \\ (\mathcal{H}x)_L \end{bmatrix}$	splitting of $\mathcal{H}x$ w.r.t. the dimensions of L	992
$\mathcal{X}_0$	Hilbert space; pivot space of a quasi Gelfand triple	977
$\tilde{D}_+$	dense subspace of $\mathcal{X}_0$ with an alternative inner product	977
D_-	$\left\{ g \in \mathcal{X}_0 : \sup_{g \in \tilde{D}_+ \setminus \{0\}} \frac{\lvert\langle g, f\rangle_{\mathcal{X}_0}\rvert}{\lVert f \rVert_{\mathcal{X}_+}} < +\infty \right\}$	978

2. **Boundary triple.** In this section we state the most important properties of boundary triples for skew-symmetric operators for this work. More details can be found in [6, chapter 3] and [9].

A linear relation T from a vector space X to a vector space Y is a linear subspace of $X \times Y$. Clearly, every linear operator is also a linear relation (we do not distinguish between a function and its graph). We will use the following notation

$$\ker T := \{x \in X : (x, 0) \in T\}, \qquad \operatorname{ran} T := \{y \in Y : \exists x : (x, y) \in T\},$$
$$\operatorname{mul} T := \{y \in Y : (0, y) \in T\}, \qquad \operatorname{dom} T := \{x \in X : \exists y : (x, y) \in T\}.$$

Thus, T is single-valued, if $\operatorname{mul} T = \{0\}$. The closure $\overline{T}$ of a linear relation T is the closure in $X \times Y$. Note that every linear relation is closable. Also every operator has a closure as a linear relation, but its closure can be multi-valued. Therefore, showing $\operatorname{mul} \overline{T} = \{0\}$ is necessary, even if $\operatorname{mul} T = \{0\}$. For an additional linear relation S from Y to another vector space Z we define the composition ST as

$$ST := \{(x, z) \in X \times Z : \exists y \in Y \text{ such that } (x, y) \in T \text{ and } (y, z) \in S\}.$$

For a linear relation T from a Hilbert space X to a Hilbert space Y the adjoint relation is defined by

$$T^* := \{(u, v) \in Y \times X : \langle u, y\rangle_Y = \langle v, x\rangle_X \text{ for all } (x, y) \in T\}$$

and the following holds true

$$\ker T^* = (\operatorname{ran} T)^\perp, \quad \operatorname{mul} T^* = (\operatorname{dom} T)^\perp \quad \text{and} \quad T^* = \begin{bmatrix} 0 & \mathrm{I}_Y \\ -\mathrm{I}_X & 0 \end{bmatrix} T^\perp,$$

where $\begin{bmatrix} 0 & \mathrm{I}_Y \\ -\mathrm{I}_X & 0 \end{bmatrix} T := \{(y, -x) : (x, y) \in T\}$ and $T^\perp$ is the orthogonal complement in $X \times Y$. A linear relation T on a Hilbert space H (from H to H) is *dissipative*, if $\operatorname{Re}\langle x, y\rangle_H \leq 0$ for every $(x, y) \in T$ and *maximal dissipative*, if additionally there is no proper dissipative extension of T. The linear relation T is *(maximal) accretive*, if $-T := \{(x, -y) : (x, y) \in T\}$ is (maximal) dissipative. More details can be found in [4].

Definition 2.1. Let A_0 be a densely defined, skew-symmetric, and closed operator on a Hilbert space X. By a *boundary triple* for A_0^* we mean a triple $(\mathcal{B}, B_1, B_2)$ consisting of a Hilbert space $\mathcal{B}$, and two linear operators $B_1, B_2 \colon \operatorname{dom} A_0^* \to \mathcal{B}$ such that

(i) the mapping $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \colon \operatorname{dom} A_0^* \to \mathcal{B} \times \mathcal{B}, x \mapsto \begin{bmatrix} B_1 x \\ B_2 x \end{bmatrix}$ is surjective, and

(ii) for $x, y \in \operatorname{dom} A_0^*$ there holds

$$\langle A_0^* x, y\rangle_X + \langle x, A_0^* y\rangle_X = \langle B_1 x, B_2 y\rangle_{\mathcal{B}} + \langle B_2 x, B_1 y\rangle_{\mathcal{B}}. \tag{2.1}$$

The operator A_0 can be recovered by restricting $-A_0^*$ to $\ker B_1 \cap \ker B_2$ as the next lemma will show. However, if A_0^* satisfied item (i) and item (ii) but wasn't the adjoint of a skew-symmetric operator, then the next lemma would not hold as Example A.1 demonstrates. Consequently, Proposition 2.3 would also not hold. This should highlight the importance of A_0^* being the adjoint of a skew-symmetric operator in the definition of a boundary triple.

Lemma 2.2. *Let A_0 be a densely defined, skew-symmetric, and closed operator on a Hilbert space X and $(\mathcal{B}, B_1, B_2)$ be a boundary triple for A_0^*. Then $A_0 = -A_0^*|_{\ker B_1 \cap \ker B_2}$.*

A proof can be found in [6, p. 155]. The following result is Theorem 2.2 from [9].

Proposition 2.3. *Let A_0 be a skew-symmetric operator and $(\mathcal{B}, B_1, B_2)$ be a boundary triple for A_0^*. Consider the restriction A of A_0^* to a subspace $\mathcal{D}$ containing $\ker B_1 \cap \ker B_2$. Define a subspace of $\mathcal{B} \times \mathcal{B}$ by $\mathcal{C} := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \mathcal{D}$. Then the following claims are true:*

(i) *The domain of A can be written as*

$$\operatorname{dom} A = \mathcal{D} = \left\{ d \in \operatorname{dom} A_0^* : \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} d \in \mathcal{C} \right\}.$$

(ii) *The operator closure of A is A_0^* restricted to*

$$\tilde{\mathcal{D}} := \left\{ d \in \operatorname{dom} A_0^* : \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} d \in \overline{\mathcal{C}} \right\},$$

where $\overline{\mathcal{C}}$ is the closure in $\mathcal{B}^2$. Therefore, A is closed if and only if $\mathcal{C}$ is closed.

(iii) *The adjoint A^* is the restriction of $-A_0^*$ to $\mathcal{D}'$, where*

$$\mathcal{D}' := \left\{ d' \in \operatorname{dom} A_0^* : \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} d' \in \underbrace{\begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix} \mathcal{C}^\perp}_{=-\mathcal{C}^*} \right\}.$$

(iv) *The operator A is (maximal) dissipative if and only if $\mathcal{C}$ is a (maximal) dissipative relation. It also holds that A is (maximal) accretive, if and only if $\mathcal{C}$ is (maximal) accretive.*

3. **Differential operators.** Before we start analyzing port-Hamiltonian systems we will make some observation about the differential operators that will appear in the PDE. In this section we take care of all the technical details of these differential operators. Since it doesn't really make a difference whether we use the scalar field $\mathbb{R}$ or $\mathbb{C}$ we will use $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ for the scalar field. The following assumption will be made for the rest of this work.

Assumption 3.1. Let $m_1, m_2, n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ be open with a bounded Lipschitz boundary, and $L = (L_i)_{i=1}^n$ such that $L_i \in \mathbb{K}^{m_1 \times m_2}$ for all $i \in \{1, \ldots, n\}$. Corresponding to L we also have $L^{\mathsf{H}} \coloneqq (L_i^{\mathsf{H}})_{i=1}^n$, where L_i^{H} denotes the complex conjugated transposed (Hermitian transposed) matrix.

We will write $\mathcal{D}(\Omega)$ for the set of all $C^\infty(\Omega)$ functions with compact support in Ω. Its dual space, the space of distributions, will be denoted by $\mathcal{D}'(\Omega)$ (details on distributions can be found in [7]). Moreover, we will write $\mathcal{D}(\mathbb{R}^n)\big|_\Omega$ for $\{f\big|_\Omega : f \in \mathcal{D}(\mathbb{R}^n)\}$. We will use ∂_i as a short notation for $\frac{\partial}{\partial \zeta_i}$. We denote the boundary trace operator by $\gamma_0 \colon H^1(\Omega, X) \to L^2(\partial\Omega, X)$ for a Banach space X.

Sometimes it can be confusing to pay attention to the antilinear structure of an inner product of a Hilbert space, when switching between the inner product and the dual pairing. Thus, for the sake of clarity we will always consider the antidual space instead of the dual space, which is the space of all continuous antilinear mappings from the topological vector space into its scalar field. Hence, both the inner product and the (anti)dual pairing is linear in one component and antilinear in the other. So also $\mathcal{D}'(\Omega)$ is actually the antidual space of $\mathcal{D}(\Omega)$.

Sometimes we will write $\langle \psi, \phi \rangle_{\mathcal{D}', \mathcal{D}}$ instead of $\langle \psi, \phi \rangle_{\mathcal{D}'(\Omega)^k, \mathcal{D}(\Omega)^k}$, if Ω and $k \in \mathbb{N}$ are clear or $\langle \psi, \phi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)}$, if only $k \in \mathbb{N}$ is clear.

Definition 3.2. Let L be as in Assumption 3.1. Then we define

$$L_\partial \coloneqq \sum_{i=1}^n \partial_i L_i \quad \text{and} \quad L_\partial^{\mathsf{H}} \coloneqq (L^{\mathsf{H}})_\partial = \sum_{i=1}^n \partial_i L_i^{\mathsf{H}}$$

as operators from $\mathcal{D}'(\Omega)^{m_2}$ to $\mathcal{D}'(\Omega)^{m_1}$ and from $\mathcal{D}'(\Omega)^{m_1}$ to $\mathcal{D}'(\Omega)^{m_2}$, respectively. Furthermore, we define the space

$$H(L_\partial, \Omega) \coloneqq \left\{ f \in L^2(\Omega, \mathbb{K}^{m_2}) : L_\partial f \in L^2(\Omega, \mathbb{K}^{m_1}) \right\}.$$

This space is endowed with the inner product

$$\langle f, g \rangle_{H(L_\partial, \Omega)} \coloneqq \langle f, g \rangle_{L^2(\Omega, \mathbb{K}^{m_2})} + \langle L_\partial f, L_\partial g \rangle_{L^2(\Omega, \mathbb{K}^{m_1})}.$$

The space $H_0(L_\partial, \Omega)$ is defined as $\overline{\mathcal{D}(\Omega)^{m_2}}^{\|\cdot\|_{H(L_\partial, \Omega)}}$. We denote the outward pointing normed normal vector on $\partial\Omega$ by ν and its i-th component by ν_i. Moreover, we define

$$L_\nu \coloneqq \sum_{i=1}^n \nu_i L_i \colon \begin{cases} L^2(\partial\Omega, \mathbb{K}^{m_2}) & \to & L^2(\partial\Omega, \mathbb{K}^{m_1}), \\ f & \mapsto & \sum_{i=1}^n \nu_i L_i f, \end{cases}$$

and $L_\nu^{\mathsf{H}} \coloneqq (L^{\mathsf{H}})_\nu$.

The operator L_∂ can also be regarded as a linear unbounded operator from $L^2(\Omega,\mathbb{K}^{m_2})$ to $L^2(\Omega,\mathbb{K}^{m_1})$ with domain $H(L_\partial,\Omega)$. In fact this is what we will do most of the time. The same goes for L_∂^{H} with domain $H(L_\partial^{\mathsf{H}},\Omega)$. Since $\nu\in L^\infty(\partial\Omega,\mathbb{R}^n)$ the mappings L_ν and L_ν^{H} are well-defined and bounded.

For convenience we will write $H^1(\Omega)^k$ instead of $H^1(\Omega,\mathbb{K}^k)$ and $L^2(\Omega)^k$ instead of $L^2(\Omega,\mathbb{K}^k)$ for $k\in\mathbb{N}$.

Clearly, $\mathcal{D}(\mathbb{R}^n)^{m_2}\big|_\Omega\subseteq H^1(\Omega)^{m_2}\subseteq H(L_\partial,\Omega)$ and $\mathcal{D}(\mathbb{R}^n)^{m_1}\big|_\Omega\subseteq H^1(\Omega)^{m_1}\subseteq H(L_\partial^{\mathsf{H}},\Omega)$.

Example 3.3. Let us regard the following matrices

$$L_1=\begin{bmatrix}1&0&0\end{bmatrix},\quad L_2=\begin{bmatrix}0&1&0\end{bmatrix},\quad\text{and}\quad L_3=\begin{bmatrix}0&0&1\end{bmatrix}.$$

Then we obtain the corresponding differential operators

$$L_\partial=\begin{bmatrix}\partial_1&\partial_2&\partial_3\end{bmatrix}=\operatorname{div}\quad\text{and}\quad L_\partial^{\mathsf{H}}=\begin{bmatrix}\partial_1\\\partial_2\\\partial_3\end{bmatrix}=\operatorname{grad}.$$

The corresponding operator L_ν that acts on $L^2(\partial\Omega)$ can be written as an inner product

$$L_\nu f=\begin{bmatrix}\nu_1&\nu_2&\nu_3\end{bmatrix}\begin{bmatrix}f_1\\f_2\\f_3\end{bmatrix}=\nu\cdot f.$$

Clearly the previous example can be extended to any finite dimension.

Example 3.4. The following matrices will construct the rotation operator.

$$L_1=\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix},\quad L_2=\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix},\quad\text{and}\quad L_3=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}.$$

In this example we have $L_i^{\mathsf{H}}=-L_i$. Furthermore, the corresponding differential operator is

$$L_\partial=\begin{bmatrix}0&-\partial_3&\partial_2\\\partial_3&0&-\partial_1\\-\partial_2&\partial_1&0\end{bmatrix}=\operatorname{rot}=-L_\partial^{\mathsf{H}}.$$

The corresponding operator L_ν that acts on $L^2(\partial\Omega)$ can be written as a cross product

$$L_\nu f=\begin{bmatrix}0&-\nu_3&\nu_2\\\nu_3&0&-\nu_1\\-\nu_2&\nu_1&0\end{bmatrix}\begin{bmatrix}f_1\\f_2\\f_3\end{bmatrix}=\nu\times f.$$

Lemma 3.5. *The operator L_∂ with* $\operatorname{dom} L_\partial=H(L_\partial,\Omega)$ *is a closed operator from $L^2(\Omega)^{m_2}$ to $L^2(\Omega)^{m_1}$ and $H(L_\partial,\Omega)$ endowed with the inner product $\langle .,.\rangle_{H(L_\partial,\Omega)}$ is a Hilbert space.*

Note that for $f\in\mathcal{D}'(\Omega)^{m_2}$ and $\phi\in\mathcal{D}(\Omega)^{m_1}$ we have

$$\begin{aligned}\langle L_\partial f,\phi\rangle_{\mathcal{D}'(\Omega)^{m_1},\mathcal{D}(\Omega)^{m_1}}&=\sum_{i=1}^n\langle\partial_i L_i f,\phi\rangle_{\mathcal{D}'(\Omega)^{m_1},\mathcal{D}(\Omega)^{m_1}}\\&=\sum_{i=1}^n\langle f,-\partial_i L_i^{\mathsf{H}}\phi\rangle_{\mathcal{D}'(\Omega)^{m_2},\mathcal{D}(\Omega)^{m_2}}=\langle f,-L_\partial^{\mathsf{H}}\phi\rangle_{\mathcal{D}'(\Omega)^{m_2},\mathcal{D}(\Omega)^{m_2}}.\end{aligned}$$

Proof. Let $\big((f_k, L_\partial f_k)\big)_{k\in\mathbb{N}}$ be a sequence in L_∂ that converges to a point $(f,g) \in L^2(\Omega)^{m_2} \times L^2(\Omega)^{m_1}$. For an arbitrary $\phi \in \mathcal{D}(\Omega)^{m_1}$ we have

$$\begin{aligned}\langle g, \phi\rangle_{\mathcal{D}'(\Omega)^{m_1},\mathcal{D}(\Omega)^{m_1}} &= \lim_{k\to\infty} \langle L_\partial f_k, \phi\rangle_{\mathcal{D}'(\Omega)^{m_1},\mathcal{D}(\Omega)^{m_1}} \\ &= \lim_{k\to\infty} \langle f_k, -L_\partial^{\mathsf{H}} \phi\rangle_{\mathcal{D}'(\Omega)^{m_1},\mathcal{D}(\Omega)^{m_1}} \\ &= \langle f, -L_\partial^{\mathsf{H}}\phi\rangle_{\mathcal{D}'(\Omega)^{m_2},\mathcal{D}(\Omega)^{m_2}} \\ &= \langle L_\partial f, \phi\rangle_{\mathcal{D}'(\Omega)^{m_1},\mathcal{D}(\Omega)^{m_1}},\end{aligned}$$

which implies $g = L_\partial f$. Since g is also in $L^2(\Omega)^{m_1}$, we conclude that L_∂ is closed. Hence, $\operatorname{dom} L_\partial = H(L_\partial, \Omega)$ endowed with the graph norm of L_∂, which is induced by $\langle .,.\rangle_{H(L_\partial,\Omega)}$, is a Hilbert space. ❑

Lemma 3.6. *The adjoint of L_∂ with $\operatorname{dom} L_\partial = H(L_\partial, \Omega)$ (as an unbounded operator/linear relation from $L^2(\Omega)^{m_2}$ to $L^2(\Omega)^{m_1}$) is given by $L_\partial^* g = -L_\partial^{\mathsf{H}} g$ for $g \in \operatorname{dom} L_\partial^* \subseteq H(L_\partial^{\mathsf{H}}, \Omega)$, i.e. $L_\partial^* \subseteq -L_\partial^{\mathsf{H}}$.*

Proof. For an arbitrary $g \in \operatorname{dom} L_\partial^*$ and an arbitrary $\phi \in \mathcal{D}(\Omega)^{m_2}$ we have

$$\langle L_\partial^* g, \phi\rangle_{\mathcal{D}',\mathcal{D}} = \langle L_\partial^* g, \phi\rangle_{L^2} = \langle g, L_\partial \phi\rangle_{L^2} = \langle g, L_\partial\phi\rangle_{\mathcal{D}',\mathcal{D}} = \langle -L_\partial^{\mathsf{H}} g, \phi\rangle_{\mathcal{D}',\mathcal{D}}.$$

Therefore, $L_\partial^* g = -L_\partial^{\mathsf{H}} g$ and $L_\partial^* g \in L^2(\Omega)^{m_2}$ implies $L_\partial^{\mathsf{H}} g \in L^2(\Omega)^{m_2}$. Consequently, $\operatorname{dom} L_\partial^* \subseteq H(L_\partial^{\mathsf{H}}, \Omega)$. ❑

Remark 3.7. If L contains only Hermitian matrices ($L_i^{\mathsf{H}} = L_i$), then $L_\partial^{\mathsf{H}} = L_\partial$ and L_∂^* is skew-symmetric by the previous lemma.

The next result is an integration by parts version for L_∂. This will be helpful to construct a boundary triple for the differential operator in the port-Hamiltonian PDE.

Lemma 3.8. *Let $f \in H^1(\Omega)^{m_2}$ and $g \in H^1(\Omega)^{m_1}$. Then we have*

$$\begin{aligned}\langle L_\partial f, g\rangle_{L^2(\Omega)^{m_1}} + \langle f, L_\partial^{\mathsf{H}} g\rangle_{L^2(\Omega)^{m_2}} &= \langle L_\nu \gamma_0 f, \gamma_0 g\rangle_{L^2(\partial\Omega)^{m_1}} \\ &= \langle \gamma_0 f, L_\nu^{\mathsf{H}} \gamma_0 g\rangle_{L^2(\partial\Omega)^{m_2}}.\end{aligned} \tag{3.1}$$

Proof. Let $f \in \mathcal{D}(\mathbb{R}^n)^{m_2}\big|_\Omega$ and $g \in \mathcal{D}(\mathbb{R}^n)^{m_1}\big|_\Omega$. By the definition of L_∂ and L_∂^{H}, and the linearity of the scalar product we can write the left-hand-side of (3.1) as

$$\int_\Omega \sum_{i=1}^n \langle \partial_i L_i f, g\rangle + \langle f, \partial_i L_i^{\mathsf{H}} g\rangle \,\mathrm{d}\lambda = \int_\Omega \sum_{i=1}^n \langle \partial_i L_i f, g\rangle + \langle L_i f, \partial_i g\rangle \,\mathrm{d}\lambda,$$

where λ denotes the Lebesgue measure. By the product rule for derivatives and Gauß's theorem (divergence theorem) (see [7, eq. (3.1.6)] or [13, Remark 13.7.2]) this is equal to

$$\int_\Omega \sum_{i=1}^n \partial_i \langle L_i f, g\rangle \,\mathrm{d}\lambda = \int_{\partial\Omega} \sum_{i=1}^n \nu_i \gamma_0 \langle L_i f, g\rangle \,\mathrm{d}\mu = \int_{\partial\Omega} \langle L_\nu \gamma_0 f, \gamma_0 g\rangle \,\mathrm{d}\mu,$$

where ν denotes the outward pointing normed normal vector on $\partial\Omega$ and μ denotes the surface measure of $\partial\Omega$. By density we can extend this equality for $f \in H^1(\Omega)^{m_2}$ and $g \in H^1(\Omega)^{m_1}$. ❑

Corollary 3.9. *Let $f \in H^1(\Omega)^{m_2}$ and $g \in H^1(\Omega)^{m_1}$. Then we have*

$$\left|\langle L_\nu \gamma_0 f, \gamma_0 g\rangle_{L^2(\partial\Omega)^{m_1}}\right| \le \|f\|_{H(L_\partial,\Omega)} \|g\|_{H(L_\partial^{\mathsf{H}},\Omega)}.$$

Proof. Lemma 3.8, the triangular inequality and Cauchy Schwarz's inequality yield

$$\begin{aligned}\left|\langle L_\nu \gamma_0 f, \gamma_0 g\rangle_{L^2(\partial\Omega)^{m_1}}\right| &\leq \left|\langle L_\partial f, g\rangle_{L^2(\Omega)^{m_1}}\right| + \left|\langle f, L_\partial^{\mathsf{H}} g\rangle_{L^2(\Omega)^{m_2}}\right| \\ &\leq \|L_\partial f\|_{L^2(\Omega)^{m_1}}\|g\|_{L^2(\Omega)^{m_1}} + \|f\|_{L^2(\Omega)^{m_2}}\|L_\partial^{\mathsf{H}} g\|_{L^2(\Omega)^{m_2}} \\ &\leq \sqrt{\|L_\partial f\|_{L^2}^2 + \|f\|_{L^2}^2}\sqrt{\|g\|_{L^2}^2 + \|L_\partial^{\mathsf{H}} g\|_{L^2}^2} \\ &= \|f\|_{H(L_\partial,\Omega)}\|g\|_{H(L_\partial^{\mathsf{H}},\Omega)}.\end{aligned}$$

❑

Note that $\Omega = \mathbb{R}^n$ satisfies the assumptions in Assumption 3.1. Hence, all the previous results hold true for $\Omega = \mathbb{R}^n$.

Our next goal is to show that $\mathcal{D}(\mathbb{R}^n)^{m_2}\big|_\Omega$ is dense in $H(L_\partial, \Omega)$; see Theorem 3.18. In order to archive this we will present some regularization and continuity results. In particular the density is needed to extend the integration by parts formula (Lemma 3.8) for $f \in H(L_\partial, \Omega)$ and $g \in H(L_\partial^{\mathsf{H}}, \Omega)$.

Lemma 3.10. *The mapping* $\iota\colon H(L_\partial, \mathbb{R}^n) \to H(L_\partial, \Omega)$, $f \mapsto f\big|_\Omega$ *is well-defined and continuous for any open set* $\Omega \subseteq \mathbb{R}^n$*. In particular,* $L_\partial(f\big|_\Omega) = (L_\partial f)\big|_\Omega$*. Moreover, if* $f_k \to f$ *in* $H(L_\partial, \mathbb{R}^n)$*, then* $f_k \to f$ *in* $H(L_\partial, \Omega)$*.*

Hence, we can always regard an $f \in H(L_\partial, \mathbb{R}^n)$ as an element of $H(L_\partial, \Omega)$, especially when $\operatorname{supp} f \subseteq \overline{\Omega}$ – then it is also possible to recover f from $f\big|_\Omega$.

Proof. If $f \in H(L_\partial, \mathbb{R}^n)$, then $f \in L^2(\mathbb{R}^n)^{m_2}$ and $L_\partial f \in L^2(\mathbb{R}^n)^{m_1}$. Hence, it is easy to see that $\|f\big|_\Omega\|_{L^2(\Omega)} \leq \|f\|_{L^2(\mathbb{R}^n)}$ and $\|(L_\partial f)\big|_\Omega\|_{L^2(\Omega)} \leq \|L_\partial f\|_{L^2(\mathbb{R}^n)}$. Note that $\mathcal{D}(\Omega) \subseteq \mathcal{D}(\mathbb{R}^n)$, and that for $g \in L^2(\mathbb{R}^n)$ and $\phi \in \mathcal{D}(\Omega)$

$$\langle g, \phi\rangle_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)} = \langle g, \phi\rangle_{L^2(\mathbb{R}^n)} = \big\langle g\big|_\Omega, \phi\big\rangle_{L^2(\Omega)} = \big\langle g\big|_\Omega, \phi\big\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}.$$

Hence, for $f \in H(L_\partial, \mathbb{R}^n)$ and $\phi \in \mathcal{D}(\Omega)$ we have

$$\begin{aligned}\big\langle L_\partial(f\big|_\Omega), \phi\big\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} &= \big\langle f\big|_\Omega, -L_\partial^{\mathsf{H}}\phi\big\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \langle f, -L_\partial^{\mathsf{H}}\phi\rangle_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)} \\ &= \langle L_\partial f, \phi\rangle_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)} \\ &= \big\langle (L_\partial f)\big|_\Omega, \phi\big\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)},\end{aligned}$$

which implies $L_\partial(f\big|_\Omega) = (L_\partial f)\big|_\Omega$ in $\mathcal{D}'(\Omega)$. Since the latter is in $L^2(\Omega)$, we conclude $f\big|_\Omega \in H(L_\partial, \Omega)$. Consequently, ι is well-defined and $\|\iota f\|_{H(L_\partial,\Omega)} \leq \|f\|_{H(L_\partial,\mathbb{R}^n)}$ by the norm estimates from the beginning. Since ι is linear this implies the continuity of ι and in turn the last assertion of the lemma. ❑

Lemma 3.11. *Let* $D_\eta\colon L^2(\mathbb{R}^n)^k \to L^2(\mathbb{R}^n)^k$ *be the mapping defined by* $(D_\eta f)(\zeta) \coloneqq f(\eta\zeta)$*, where* $\eta \in (0, +\infty)$ *and* $k \in \mathbb{N}$*. Then* D_η *converges in the strong operator topology to* I *for* $\eta \to 1$*.*

Proof. For $\phi \in \mathcal{D}(\mathbb{R}^n)^k$ we will show that $\eta \mapsto D_\eta\phi$ from $(0, +\infty)$ to $L^2(\mathbb{R}^n)^k$ is continuous:

$$\begin{aligned}\|D_{\eta_1}\phi - D_{\eta_2}\phi\|_{L^2}^2 &= \int_{\mathbb{R}^n} \|\phi(\eta_1\zeta) - \phi(\eta_2\zeta)\|_{\mathbb{K}^k}^2 \,\mathrm{d}\lambda(\zeta) \\ &= \frac{1}{\eta_2^{2n}}\int_{\mathbb{R}^n} \Big\|\phi\big(\frac{\eta_1}{\eta_2}\zeta\big) - \phi(\zeta)\Big\|_{\mathbb{K}^k}^2 \,\mathrm{d}\lambda(\zeta) \to 0 \quad \text{for} \quad \eta_2 \to \eta_1\end{aligned}$$

by Lebesgue's dominated convergence theorem, where λ denotes the Lebesgue measure. For $f \in L^2(\mathbb{R}^n)^k$ there exists a sequence $(\phi_m)_{m\in\mathbb{N}}$ of $\mathcal{D}(\mathbb{R}^n)^k$ functions that converges to f (w.r.t. $\|.\|_{L^2}$). Hence,

$$\|D_\eta\phi_m - D_\eta f\|_{L^2} = \frac{1}{\eta^n}\|\phi_m - f\|_{L^2}$$

and $D_\eta\phi_m$ converges uniformly in $\eta \in (\epsilon, +\infty), \epsilon > 0$ to $D_\eta f$ for $m \to \infty$. Consequently $\eta \mapsto D_\eta f$ is also continuous from $(\epsilon, +\infty)$ to $L^2(\Omega)^k$ and in particular $D_\eta f \to f$ for $\eta \to 1$. ❑

Definition 3.12. A set $O \subseteq \mathbb{R}^n$ is *strongly star-shaped with respect to* ζ_0, if for every $\zeta \in \overline{O}$ the half-open line segment $\{\theta(\zeta-\zeta_0)+\zeta_0 : \theta \in [0,1)\}$ is contained in O. We call O *strongly star-shaped*, if there is a ζ_0 such that O is strongly star-shaped with respect to ζ_0.

Note that this is equivalent to

$$\theta(\overline{O} - \zeta_0) + \zeta_0 \subseteq O \quad \text{for all} \quad \theta \in [0,1).$$

Lemma 3.13. *Let $f \in H(L_\partial, \mathbb{R}^n)$ and $\zeta_0 \in \mathbb{R}^n$. Furthermore, let $f_\theta(\zeta) \coloneqq f(\frac{1}{\theta}(\zeta - \zeta_0) + \zeta_0)$ for $\theta \in (0,1)$ and a.e. $\zeta \in \mathbb{R}^n$. Then $f_\theta \in H(L_\partial, \mathbb{R}^n)$ and $f_\theta \to f$ in $H(L_\partial, \mathbb{R}^n)$ as $\theta \to 1$. If there exists a strongly star-shaped set O with respect to the previous ζ_0 such that $\operatorname{supp} f \subseteq \overline{O}$, then $\operatorname{supp} f_\theta \subseteq O$ for $\theta \in (0,1)$.*

Proof. Let $f \in H(L_\partial, \mathbb{R}^n)$ and $\alpha(\zeta) \coloneqq \frac{1}{\theta}(\zeta - \zeta_0) + \zeta_0$. Then it is easy to see that $f_\theta = f \circ \alpha$ and $f_\theta \in L^2(\mathbb{R}^n)^{m_2}$. By change of variables we have

$$\begin{aligned}
\langle L_\partial(f\circ\alpha), \phi\rangle_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)} &= \langle f, -(L_\partial^{\mathsf{H}}\phi)\circ\alpha^{-1}\theta^n\rangle_{L^2(\mathbb{R}^n)} \\
&= \Big\langle f, -\sum_{i=1}^n L_i^{\mathsf{H}}\partial_i\Big(\phi\circ\alpha^{-1}\frac{1}{\theta}\Big)\theta^n\Big\rangle_{L^2(\mathbb{R}^n)} \\
&= \Big\langle f, -L_\partial^{\mathsf{H}}\Big(\frac{1}{\theta}\phi\circ\alpha^{-1}\Big)\theta^n\Big\rangle_{L^2(\mathbb{R}^n)} = \Big\langle \frac{1}{\theta}(L_\partial f)\circ\alpha, \phi\Big\rangle_{L^2(\mathbb{R}^n)} \\
&= \Big\langle \frac{1}{\theta}(L_\partial f)\circ\alpha, \phi\Big\rangle_{\mathcal{D}'(\mathbb{R}^n),\mathcal{D}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, $L_\partial f_\theta = \frac{1}{\theta}(L_\partial f)_\theta$ and $f_\theta \in H(L_\partial, \mathbb{R}^n)$. We can also write f_θ as $T_{\zeta_0} D_{\frac{1}{\theta}} T_{-\zeta_0} f$, where $T_\xi \colon L^2(\mathbb{R}^n)^{m_2} \to L^2(\mathbb{R}^n)^{m_2}$ is the translation mapping $f \mapsto f(.+\xi)$ and $D_\eta \colon L^2(\mathbb{R}^n)^{m_2} \to L^2(\mathbb{R}^n)^{m_2}$ is the mapping from Lemma 3.11. Since T_ξ is bounded and D_η converges strongly to I as $\eta \to 1$, we conclude $f_\theta \to f$ in $L^2(\mathbb{R}^n)^{m_2}$ as $\theta \to 1$ and $L_\partial f_\theta = \frac{1}{\theta}(L_\partial f)_\theta \to L_\partial f$ in $L^2(\mathbb{R}^n)^{m_1}$ as $\theta \to 1$. Hence, $f_\theta \to f$ in $H(L_\partial, \mathbb{R}^n)$.

Let O be strongly star-shaped with respect to ζ_0 and $\operatorname{supp} f \subseteq \overline{O}$. Then for $\theta \in (0,1)$

$$\operatorname{supp} f_\theta = \theta(\operatorname{supp} f - \zeta_0) + \zeta_0 \subseteq \theta(\overline{O} - \zeta_0) + \zeta_0 \subseteq O.$$ ❑

Remark 3.14. If $f \in H(L_\partial, \Omega)$ and $\psi \in \mathcal{D}(\mathbb{R}^n)\big|_\Omega$, then by the product rule for distributional derivatives also $\psi f \in H(L_\partial, \Omega)$ and $L_\partial(\psi f) = \psi L_\partial f + \sum_{i=1}^n (\partial_i\psi)L_i f$ (see [7, equation (3.1.1)']).

Lemma 3.15. *For every $f \in H(L_\partial, \mathbb{R}^n)$ exists a sequence $(f_k)_{k\in\mathbb{N}}$ in $H(L_\partial, \mathbb{R}^n)$ with compact support $\operatorname{supp} f_k \subseteq \operatorname{supp} f$ that converges to f in $H(L_\partial, \mathbb{R}^n)$.*

Proof. Let $\psi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ be such that

$$\psi(\zeta) \in \begin{cases} \{1\}, & \text{if } \|\zeta\| \le 1, \\ [0,1], & \text{if } 1 < \|\zeta\| < 2, \\ \{0\}, & \text{if } \|\zeta\| \ge 2. \end{cases}$$

Then $f_k \coloneqq \psi(\frac{\cdot}{k})f \in L^2(\mathbb{R}^n)^{m_2}$ and $f_k \to f$ in L^2. By the previous remark we have $L_\partial f_k = \psi(\frac{\cdot}{k})L_\partial f + \frac{1}{k}\sum_{i=1}^n (\partial_i \psi)(\frac{1}{k}.)L_i f$ and therefore $f_k \in H(L_\partial, \mathbb{R}^n)$. Since $\|\partial_i \psi\|_\infty < \infty$ and $\|L_i f\|_{L^2} \le \|L_i\| \|f\|_{L^2} < \infty$, we have $L_\partial f_k \to L_\partial f$ as $\psi(\frac{\cdot}{k})L_\partial f \to L_\partial f$ in $L^2(\mathbb{R}^n)^{m_2}$ and consequently $f_k \to f$ in $H(L_\partial, \mathbb{R}^n)$. ❑

The next result is essentially [3, Proposition 2.5.4, page 69], except that we allow Ω to be unbounded.

Lemma 3.16. *For $\Omega \subseteq \mathbb{R}^n$ (open with bounded Lipschitz boundary) there exists an open covering $(O_i)_{i=0}^k$ of $\overline{\Omega}$ such that $O_i \cap \Omega$ is bounded and strongly star-shaped for $i \in \{1, \dots, k\}$ and $\overline{O_0} \subseteq \Omega$*

Proof. Since Ω has a bounded Lipschitz boundary, there is an open ball $B_r(0)$ such that $\partial\Omega \subseteq B_r(0)$. Hence, $B_r(0) \cap \Omega$ is bounded and open with bounded Lipschitz boundary and we can apply [3, Proposition 2.5.4, page 69]. This gives an open covering $(O_i)_{i=1}^k$ of $B_r(0) \cap \Omega$ and in particular of $\partial\Omega$ such that $O_i \cap \Omega$ is strongly star-shaped. We define O_0 as $B_\epsilon(\Omega \setminus \bigcup_{i=1}^k O_i)$, where $\epsilon > 0$ is small enough such that $\overline{O_0} \subseteq \Omega$. ❑

The next lemma is similar to [5, Lemma 1, page 206], which proves the result for $L_\partial = \operatorname{rot}$. The main idea of the proof can be adopted.

Lemma 3.17. *If $f \in H(L_\partial, \Omega)$ is such that*

$$\langle L_\partial f, \phi \rangle_{L^2(\Omega)} + \langle f, L_\partial^{\mathsf{H}} \phi \rangle_{L^2(\Omega)} = 0 \quad \text{for all} \quad \phi \in \mathcal{D}(\mathbb{R}^n)^{m_1}, \tag{3.2}$$

then $f \in H_0(L_\partial, \Omega)$.

Recall the definition of a positive mollifier: Let $\rho \in \mathcal{D}(\mathbb{R}^n)$. Then we define ρ_ϵ by $\rho_\epsilon(\zeta) = \epsilon^{-n}\rho(\frac{\zeta}{\epsilon})$. We say that ρ_ϵ is a positive mollifier, if $\rho(\zeta) \ge 0$, $\int_{\mathbb{R}^n} \rho(\zeta)\, \mathrm{d}\zeta = 1$ and $\lim_{\epsilon \to 0} \rho_\epsilon = \delta_0$ in the sense of distributions, where δ_0 is the Dirac delta function ($\langle \delta_0, \phi \rangle_{\mathcal{D}', \mathcal{D}} = \phi(0)$).

In particular, for every $f \in L^2(\mathbb{R}^n)$ holds

$$\rho_\epsilon * f \coloneqq \int_{\mathbb{R}^n} \rho_\epsilon(\zeta) f(. - \zeta)\, \mathrm{d}\zeta \overset{\epsilon \to 0}{\longrightarrow} f \quad \text{in} \quad L^2(\mathbb{R}^n).$$

Proof. Let $f \in H(L_\partial, \Omega)$ satisfy (3.2). Then we have to find a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\Omega)^{m_2}$ that converges to f with respect to $\|.\|_{H(L_\partial, \Omega)}$.

We define $\tilde{f}$ and $\widetilde{L_\partial f}$ as the extension of f and $L_\partial f$ respectively on $\mathbb{R}^n$ such that these functions are 0 outside of Ω. By

$$\begin{aligned} \langle \widetilde{L_\partial f}, \phi \rangle_{\mathcal{D}'(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)} &= \langle \widetilde{L_\partial f}, \phi \rangle_{L^2(\mathbb{R}^n)} = \langle L_\partial f, \phi \rangle_{L^2(\Omega)} \overset{(3.2)}{=} \langle f, -L_\partial^{\mathsf{H}} \phi \rangle_{L^2(\Omega)} \\ &= \langle \tilde{f}, -L_\partial^{\mathsf{H}} \phi \rangle_{L^2(\mathbb{R}^n)} = \langle \tilde{f}, -L_\partial^{\mathsf{H}} \phi \rangle_{\mathcal{D}'(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)} \end{aligned}$$

for $\phi \in \mathcal{D}(\mathbb{R}^n)^{m_1}$, we see that $\widetilde{L_\partial f} = L_\partial \tilde{f}$ and $\tilde{f} \in H(L_\partial, \mathbb{R}^n)$ with $\operatorname{supp} \tilde{f} \subseteq \overline{\Omega}$.

By Lemma 3.16 there is a finite open covering $(O_i)_{i=0}^k$ of $\overline{\Omega}$ such that $O_i \cap \overline{\Omega}$ is strongly star-shaped for $i \in \{1, \dots, k\}$ and $\overline{O_0} \subseteq \Omega$. We employ a partition of unity and obtain $(\alpha_i)_{i=0}^k$, subordinate to this covering, that is

$$\alpha_i \in C^\infty(\mathbb{R}^n), \quad \operatorname{supp} \alpha_i \subseteq O_i, \quad \alpha_i(\zeta) \in [0,1], \quad \text{and} \quad \sum_{i=0}^k \alpha_i(\zeta) = 1 \quad \text{for} \quad \zeta \in \Omega.$$

Hence, $\tilde{f} = \sum_{i=0}^k \alpha_i \tilde{f}$ and we define $f_i := \alpha_i \tilde{f}$. By construction $f_i \in H(L_\partial, \mathbb{R}^n)$ and $\operatorname{supp} f_i \subseteq \overline{O_i \cap \Omega}$.

- For $i \in \{1, \dots, k\}$ we have $O_i \cap \Omega$ is strongly star-shaped. Lemma 3.13 ensures that $\operatorname{supp}(f_i)_\theta \subseteq O_i \cap \Omega$ for $\theta \in (0,1)$ and $(f_i)_\theta \to f_i$ in $H(L_\partial, \mathbb{R}^n)$ for $\theta \to 1$.

  Let ρ_ϵ be a positive mollifier. Then $\rho_\epsilon * g \to g$ in $L^2(\mathbb{R}^n)$ for an arbitrary $g \in L^2(\mathbb{R}^n)$. Since $L_\partial(\rho_\epsilon * h) = \rho_\epsilon * L_\partial h$, we also have $\rho_\epsilon * h \to h$ in $H(L_\partial, \mathbb{R}^n)$ for $h \in H(L_\partial, \mathbb{R}^n)$ and since $\rho_\epsilon \in C^\infty(\mathbb{R}^n)$ we have $\rho_\epsilon * h \in C^\infty(\mathbb{R}^n)^{m_2}$.

  For fixed $\theta \in (0,1)$ and ϵ sufficiently small, we can say $\operatorname{supp} \rho_\epsilon * (f_i)_\theta \subseteq O_i \cap \Omega$. Hence, by a diagonalization argument we find a sequence $(\rho_{\epsilon_j} * (f_i)_{\theta_j})_{j \in \mathbb{N}}$ in $\mathcal{D}(\Omega)^{m_2}$ converging to f_i in $H(L_\partial, \mathbb{R}^n)$. Doing this for every $i \in \{1, \dots, k\}$ yields sequences $(f_{i,j})_{j \in \mathbb{N}}$ in $\mathcal{D}(\Omega)^{m_2}$ converging to f_i in $H(L_\partial, \mathbb{R}^n)$.

- For f_0 we have $\operatorname{supp} f_0 \subseteq \overline{O_0} \subseteq \Omega$ and by Lemma 3.15 there exists a sequence $(g_l)_{l \in \mathbb{N}}$ in $H(L_\partial, \mathbb{R}^n)$ that converges to f_0 in $H(L_\partial, \mathbb{R}^n)$ such that every g_l has compact support in Ω. Every g_l can be approximated by $\rho_\epsilon * g_l$ for $\epsilon \to 0$ in $H(L_\partial, \mathbb{R}^n)$ and if ϵ is sufficiently small $\operatorname{supp} \rho_\epsilon * g_l \subseteq \Omega$. A diagonalization argument establishes a sequence $(f_{0,j})_{j \in \mathbb{N}}$ in $\mathcal{D}(\Omega)^{m_2}$ that converges to f_0 in $H(L_\partial, \mathbb{R}^n)$.

Consequently, $\left(\sum_{i=0}^k f_{i,j}\right)_{j \in \mathbb{N}}$ is a sequence in $\mathcal{D}(\Omega)^{m_2}$ that converges to $\tilde{f}$ in $H(L_\partial, \mathbb{R}^n)$ and by Lemma 3.10 also in $H(L_\partial, \Omega)$. ❑

Theorem 3.18. $\mathcal{D}(\mathbb{R}^n)^{m_2}\big|_\Omega$ *is dense in* $H(L_\partial, \Omega)$.

Proof. Suppose $\mathcal{D}(\mathbb{R}^n)^{m_2}\big|_\Omega$ is not dense in $H(L_\partial, \Omega)$. Then there exists a non zero $f \in H(L_\partial, \Omega)$ such that

$$\langle f, g \rangle_{H(L_\partial, \Omega)} = \langle f, g \rangle_{L^2} + \langle L_\partial f, L_\partial g \rangle_{L^2} = 0 \quad \text{for all} \quad g \in \mathcal{D}(\mathbb{R}^n)^{m_2}\big|_\Omega. \tag{3.3}$$

In particular, for an arbitrary $h \in \mathcal{D}(\Omega)^{m_2}$ we have

$$\langle f, h \rangle_{\mathcal{D}', \mathcal{D}} = \langle f, h \rangle_{L^2} = -\langle L_\partial f, L_\partial h \rangle_{L^2} = -\langle L_\partial f, L_\partial h \rangle_{\mathcal{D}', \mathcal{D}} = \langle L_\partial^{\mathsf{H}} L_\partial f, h \rangle_{\mathcal{D}', \mathcal{D}},$$

which implies that $f = L_\partial^{\mathsf{H}} L_\partial f \in L^2(\Omega)^{m_2}$ and $f_0 := L_\partial f \in H(L_\partial^{\mathsf{H}}, \Omega)$. Hence we can rewrite (3.3) as

$$\langle L_\partial^{\mathsf{H}} \underbrace{L_\partial f}_{=f_0}, g \rangle_{L^2(\Omega)} + \langle \underbrace{L_\partial f}_{=f_0}, L_\partial g \rangle_{L^2(\Omega)} = 0 \quad \text{for all} \quad g \in \mathcal{D}(\mathbb{R}^n)^{m_2}\big|_\Omega.$$

By Lemma 3.17 (switching the roles of L_∂ and L_∂^{H}) we have $f_0 \in H_0(L_\partial^{\mathsf{H}}, \Omega)$. Since $\mathcal{D}(\Omega)^{m_1}$ is dense in $H_0(L_\partial^{\mathsf{H}}, \Omega)$, there is a sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{D}(\Omega)^{m_1}$ converging to f_0 with respect to $\|.\|_{H(L_\partial^{\mathsf{H}}, \Omega)}$. The fact $f = L_\partial^{\mathsf{H}} L_\partial f = L_\partial^{\mathsf{H}} f_0$ implies

$$\begin{aligned} \langle f_0, f_n \rangle_{H(L_\partial^{\mathsf{H}}, \Omega)} &= \langle f_0, f_n \rangle_{L^2} + \langle L_\partial^{\mathsf{H}} f_0, L_\partial^{\mathsf{H}} f_n \rangle_{L^2} = \langle L_\partial f, f_n \rangle_{L^2} + \langle f, L_\partial^{\mathsf{H}} f_n \rangle_{L^2} \\ &= \langle L_\partial f, f_n \rangle_{\mathcal{D}', \mathcal{D}} - \langle L_\partial f, f_n \rangle_{\mathcal{D}', \mathcal{D}} \\ &= 0. \end{aligned}$$

Since $\|f_0\|^2_{H(L_\partial^{\mathsf{H}},\Omega)} = \lim_{n\to\infty}\langle f_0, f_n\rangle_{H(L_\partial^{\mathsf{H}},\Omega)} = 0$, we have $f_0 = 0$, which implies $f = L_\partial^{\mathsf{H}} f_0 = 0$. Hence, $\mathcal{D}(\mathbb{R}^n)^{m_2}\big|_\Omega$ is dense in $H(L_\partial,\Omega)$. ❑

4. **Port Hamiltonian systems.** In this section we will introduce linear first order port-Hamiltonian systems on multidimensional spatial domains and illustrate the difficulties we want to overcome.

Definition 4.1. Let $m \in \mathbb{N}$ and $P = (P_i)_{i=1}^n$, where P_i is a Hermitian $m\times m$ matrix. Moreover, let $\mathcal{H}\colon \Omega \to \mathbb{K}^{m\times m}$ be such that $\mathcal{H}(\zeta)^{\mathsf{H}} = \mathcal{H}(\zeta)$ and $c\mathrm{I} \le \mathcal{H}(\zeta) \le C\mathrm{I}$ for a.e. $\zeta \in \Omega$ and some constants $c, C > 0$ independent of ζ. Then we endow the space $\mathcal{X}_{\mathcal{H}} \coloneqq L^2(\Omega)^m$ with the scalar product

$$\langle f, g\rangle_{\mathcal{X}_{\mathcal{H}}} \coloneqq \frac{1}{2}\langle \mathcal{H}f, g\rangle_{L^2(\Omega)^m} = \frac{1}{2}\int_\Omega \langle \mathcal{H}(\zeta)f(\zeta), g(\zeta)\rangle_{\mathbb{K}^m}\,\mathrm{d}\lambda(\zeta).$$

We will refer to $\mathcal{X}_{\mathcal{H}}$ as the *state space* and to its elements as *state variables* or *states*. Furthermore, let $P_0 \in \mathbb{K}^{m\times m}$ be such that $P_0^{\mathsf{H}} = -P_0$. Then we will call the differential equation

$$\begin{aligned}\frac{\partial}{\partial t}x(t,\zeta) &= \sum_{i=1}^n \frac{\partial}{\partial \zeta_i} P_i\big(\mathcal{H}(\zeta)x(t,\zeta)\big) + P_0\big(\mathcal{H}(\zeta)x(t,\zeta)\big), && t\in\mathbb{R}_+, \zeta\in\Omega,\\ x(0,\zeta) &= x_0(\zeta), && \zeta\in\Omega\end{aligned} \tag{4.1}$$

a *linear, first order port-Hamiltonian system*, where $x_0 \in L^2(\Omega)^m$ is the initial state. The associated *Hamiltonian* $H : \mathcal{X}_{\mathcal{H}} \to \mathbb{R}_+ \cup \{0\}$ is defined by

$$H(x) \coloneqq \langle x, x\rangle_{\mathcal{X}_{\mathcal{H}}} = \frac{1}{2}\int_\Omega \langle \mathcal{H}(\zeta)x(\zeta), x(\zeta)\rangle_{\mathbb{K}^m}\,\mathrm{d}\lambda(\zeta),$$

where $\mathcal{H}$ is called the *Hamiltonian density*.

In most applications the Hamiltonian describes the energy in the state space.

By the convention of regarding a function $x\colon \mathbb{R}_+ \times \Omega \to \mathbb{K}^m$ as $x\colon \mathbb{R}_+ \to L^2(\Omega;\mathbb{K}^m)$ by setting $x(t) = x(t,.)$, we can rewrite the PDE (4.1) as

$$\dot{x} = \Big(\sum_{i=1}^n \partial_i P_i + P_0\Big)\mathcal{H}x = (P_\partial + P_0)\mathcal{H}x, \quad x(0) = x_0,$$

where P_∂ is defined by Definition 3.2 replacing L with P.

We want to add the following assumption on P.

Assumption 4.2. Let $m, m_1, m_2 \in \mathbb{N}$ such that $m = m_1 + m_2$ and let $L = (L_i)_{i=1}^n$ such that $L_i \in \mathbb{K}^{m_1\times m_2}$. Then we assume that $P = (P_i)_{i=1}^n$ has the block structure

$$P_i = \begin{bmatrix} 0 & L_i \\ L_i^{\mathsf{H}} & 0\end{bmatrix}.$$

Assumption 4.2 implies that P contains only Hermitian matrices. According to the block structure we split $x \in \mathbb{K}^m$ into $\left[\begin{smallmatrix} x_{L^{\mathsf{H}}} \\ x_L\end{smallmatrix}\right]$, where $x_{L^{\mathsf{H}}} = (x_i)_{i=1}^{m_1}$ and $x_L = (x_i)_{i=m_1+1}^m$. We have the identities $H(P_\partial,\Omega) = H(L_\partial^{\mathsf{H}},\Omega) \times H(L_\partial,\Omega)$,

$$P_\partial = \begin{bmatrix} 0 & L_\partial \\ L_\partial^{\mathsf{H}} & 0\end{bmatrix} \quad\text{and}\quad P_\nu = \begin{bmatrix} 0 & L_\nu \\ L_\nu^{\mathsf{H}} & 0\end{bmatrix}.$$

By Lemma 3.8 we have for $x, y \in H^1(\Omega)^m$

$$\begin{aligned}\langle P_\partial x, y\rangle_{L^2(\Omega)} + \langle x, P_\partial y\rangle_{L^2(\Omega)} \\ &= \langle P_\nu \gamma_0 x, \gamma_0 y\rangle_{L^2(\partial\Omega)} \\ &= \left\langle \begin{bmatrix} 0 & L_\nu \\ L_\nu^{\mathsf{H}} & 0 \end{bmatrix} \gamma_0 \begin{bmatrix} x_{L^{\mathsf{H}}} \\ x_L \end{bmatrix}, \gamma_0 \begin{bmatrix} y_{L^{\mathsf{H}}} \\ y_L \end{bmatrix} \right\rangle_{L^2(\partial\Omega)} \\ &= \langle L_\nu \gamma_0 x_L, \gamma_0 y_{L^{\mathsf{H}}}\rangle_{L^2(\partial\Omega)} + \langle L_\nu^{\mathsf{H}} \gamma_0 x_{L^{\mathsf{H}}}, \gamma_0 y_L\rangle_{L^2(\partial\Omega)} \\ &= \langle L_\nu \gamma_0 x_L, \gamma_0 y_{L^{\mathsf{H}}}\rangle_{L^2(\partial\Omega)} + \langle \gamma_0 x_{L^{\mathsf{H}}}, L_\nu \gamma_0 y_L\rangle_{L^2(\partial\Omega)}.\end{aligned} \tag{4.2}$$

Hence, $\mathcal{B} = L^2(\partial\Omega)^{m_1}$, $B_1 x = L_\nu \gamma_0 x_L$ and $B_2 x = \gamma_0 x_{L^{\mathsf{H}}}$ is reminiscent of a boundary triple for $A_0^* = P_\partial$ ($A_0 = P_\partial^*$ is skew-symmetric by Remark 3.7). However, we need to extend (4.2) for $x, y \in H(P_\partial, \Omega)$. In order to do this we have to introduce a new norm on $L^2(\partial\Omega)^{m_1}$, which will lead to the notion of quasi Gelfand triples.

5. **Quasi Gelfand triples.**

Normally when we talk about Gelfand triples we have a Hilbert space $\mathcal{X}_0$ and another Hilbert space $\mathcal{X}_+$ that can be continuously and densely embedded into $\mathcal{X}_0$. The third space $\mathcal{X}_-$ is given by the completion of $\mathcal{X}_0$ with respect to

$$\|g\|_{\mathcal{X}_-} := \sup_{f \in \mathcal{X}_+ \setminus \{0\}} \frac{|\langle g, f\rangle|_{\mathcal{X}_0}}{\|f\|_{\mathcal{X}_+}}.$$

The duality between $\mathcal{X}_+$ and $\mathcal{X}_-$ is given by

$$\langle g, f\rangle_{\mathcal{X}_-, \mathcal{X}_+} = \lim_{k\to\infty} \langle g_k, f\rangle_{\mathcal{X}_0},$$

where $(g_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{X}_0$ that converges to g in $\mathcal{X}_-$. Details for "ordinary" Gelfand triple can be found in [6, ch. 2.1, p. 54] or in [13, ch. 2.9, p. 56].

We want to weaken the assumptions such that the norm of $\mathcal{X}_+$ is not necessarily related to the norm of $\mathcal{X}_0$. This is in particular necessary for Maxwell's equations. In Example 8.10 we point out that is not possible to associate an "ordinary" Gelfand triple to the spatial differential operator of Maxwell's equations.

We will have the following setting: Let $(\mathcal{X}_0, \langle .,.\rangle_{\mathcal{X}_0})$ be a Hilbert space and $\langle .,.\rangle_{\mathcal{X}_+}$ another inner product (not necessarily related to $\langle .,.\rangle_{\mathcal{X}_0}$) which is defined on a dense (w.r.t. $\|.\|_{\mathcal{X}_0}$) subspace $\tilde{D}_+$ of $\mathcal{X}_0$. We denote the completion of $\tilde{D}_+$ w.r.t. $\|.\|_{\mathcal{X}_+} = \sqrt{\langle .,.\rangle_{\mathcal{X}_+}}$ by $\mathcal{X}_+$. This completion is again a Hilbert space with the extension of $\langle .,.\rangle_{\mathcal{X}_+}$, for which we use the same symbol. Now we have $\tilde{D}_+$ is dense in $\mathcal{X}_0$ w.r.t. $\|.\|_{\mathcal{X}_0}$ and dense in $\mathcal{X}_+$ w.r.t. $\|.\|_{\mathcal{X}_+}$.

Summarized:

- $\mathcal{X}_0$ Hilbert space endowed with $\langle .,.\rangle_{\mathcal{X}_0}$.
- $\tilde{D}_+$ dense subspace of $\mathcal{X}_0$ (w.r.t. $\|.\|_{\mathcal{X}_0}$).
- $\langle .,.\rangle_{\mathcal{X}_+}$ another inner product defined on $\tilde{D}_+$.
- $\mathcal{X}_+$ completion of $\tilde{D}_+$ with respect to $\|.\|_{\mathcal{X}_+}$.

Example 5.1. Let $\mathcal{X}_0 = \ell^2(\mathbb{Z} \setminus \{0\})$ with the standard inner product $\langle x, y\rangle_{\mathcal{X}_0} = \sum_{n=1}^\infty x_n \overline{y_n} + x_{-n}\overline{y_{-n}}$. We define the inner product

$$\langle x, y\rangle_{\mathcal{X}_+} := \sum_{n=1}^\infty n^2 x_n \overline{y_n} + \frac{1}{n^2} x_{-n} \overline{y_{-n}}$$

and the set $\tilde{D}_+ := \{f \in \mathcal{X}_0 : \|f\|_{\mathcal{X}_+} < +\infty\}$. Clearly, this inner product is well-defined on $\tilde{D}_+$. Let e_i denote the sequence which is 1 on the i-th position and 0 elsewhere. Since $\{e_i : i \in \mathbb{Z} \setminus \{0\}\}$ is a orthonormal basis of $\mathcal{X}_0$ and contained in $\tilde{D}_+$, $\tilde{D}_+$ is dense in $\mathcal{X}_0$. The sequence $\left(\sum_{i=1}^n e_{-i}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|.\|_{\mathcal{X}_+}$ but not with respect to $\|.\|_{\mathcal{X}_0}$.

Definition 5.2. We define

$$\|g\|_{\mathcal{X}_-} := \sup_{f\in\tilde{D}_+\setminus\{0\}} \frac{|\langle g, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} \quad \text{for } g \in \mathcal{X}_0 \quad \text{and} \quad D_- := \Big\{g \in \mathcal{X}_0 : \|g\|_{\mathcal{X}_-} < +\infty\Big\}.$$

We denote the completion of D_- w.r.t. $\|.\|_{\mathcal{X}_-}$ by $\mathcal{X}_-$. We will also denote the extension of $\|.\|_{\mathcal{X}_-}$ to $\mathcal{X}_-$ by $\|.\|_{\mathcal{X}_-}$.

Remark 5.3. By definition of D_- we can identify every $g \in D_-$ with an element of $\mathcal{X}_+'$ by the continuous extension of $f \in \tilde{D}_+ \mapsto \langle g, f\rangle_{\mathcal{X}_0}$ to $\mathcal{X}_+$. The completion $\mathcal{X}_-$ is isomorphic to the closure of D_- in $\mathcal{X}_+'$ as $\|g\|_{\mathcal{X}_+'} = \|g\|_{\mathcal{X}_-}$ for $g \in D_-$.

Lemma 5.4. *D_- is complete with respect to* $\|g\|_{\mathcal{X}_-\cap\mathcal{X}_0} := \sqrt{\|g\|^2_{\mathcal{X}_0} + \|g\|^2_{\mathcal{X}_-}}$.

Proof. Let $(g_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in D_- with respect to $\|.\|_{\mathcal{X}_-\cap\mathcal{X}_0}$. Then $(g_n)_{n\in\mathbb{N}}$ is a convergent sequence in $\mathcal{X}_0$ (w.r.t. $\|.\|_{\mathcal{X}_0}$) and a Cauchy sequence in D_- (w.r.t. $\|.\|_{\mathcal{X}_-}$). We denote the limit in $\mathcal{X}_0$ by g_0. By definition of $\|.\|_{\mathcal{X}_-}$ we obtain for $f \in \tilde{D}_+$

$$|\langle g_0, f\rangle_{\mathcal{X}_0}| = \lim_{n\to\infty} |\langle g_n, f\rangle_{\mathcal{X}_0}| \le \lim_{n\to\infty} \|g_n\|_{\mathcal{X}_-}\|f\|_{\mathcal{X}_+} \le C\|f\|_{\mathcal{X}_+}$$

and consequently $g_0 \in D_-$.

Let $\epsilon > 0$ be arbitrary. Since $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|.\|_{\mathcal{X}_-}$, there is an $n_0 \in \mathbb{N}$ such that for all $f \in \tilde{D}_+$ with $\|f\|_{\mathcal{X}_+} = 1$

$$|\langle g_n - g_m, f\rangle_{\mathcal{X}_0}| \le \frac{\epsilon}{2}, \quad \text{if} \quad n, m \ge n_0$$

holds true. Furthermore, for every $f \in \tilde{D}_+$ there exists an $m_f \ge n_0$ such that $|\langle g_0 - g_{m_f}, f\rangle_{\mathcal{X}_0}| \le \frac{\epsilon\|f\|_{\mathcal{X}_+}}{2}$, because $g_m \to g_0$ w.r.t. $\|.\|_{\mathcal{X}_0}$. This yields

$$\frac{|\langle g_0 - g_n, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} \le \frac{|\langle g_0 - g_{m_f}, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} + \frac{|\langle g_{m_f} - g_n, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} \le \epsilon, \quad \text{if} \quad n \ge n_0.$$

Since the right-hand-side is independent of f, we obtain

$$\|g_0 - g_n\|_{\mathcal{X}_-} = \sup_{f\in\tilde{D}_+\setminus\{0\}} \frac{|\langle g_0 - g_n, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} \le \epsilon, \quad \text{if} \quad n \ge n_0.$$

Hence, g_0 is also the limit of $(g_n)_{n\in\mathbb{N}}$ with respect to $\|.\|_{\mathcal{X}_-}$ and consequently the limit of $(g_n)_{n\in\mathbb{N}}$ with respect to $\|.\|_{\mathcal{X}_-\cap\mathcal{X}_0}$. ❑

Lemma 5.5. *The embedding* $\tilde{\iota}_+ : \tilde{D}_+ \subseteq \mathcal{X}_+ \to \mathcal{X}_0, f \mapsto f$ *is a densely defined operator with* $\operatorname{ran} \tilde{\iota}_+$ *is dense in* $\mathcal{X}_0$ *and* $\ker \tilde{\iota}_+ = \{0\}$*. Furthermore, the embedding* $\iota_- : D_- \subseteq \mathcal{X}_- \to \mathcal{X}_0, g \mapsto g$ *is closed and* $\ker \iota_- = \{0\}$.

Proof. By assumption on $\tilde{D}_+$ and definition of $\mathcal{X}_+$ the embedding $\tilde{\iota}_+$ is densely defined and has a dense range. Clearly, $\ker \tilde{\iota}_+ = \{0\}$ and $\ker \iota_- = \{0\}$. By Lemma 5.4 ι_- is closed. ❑

Lemma 5.6. *Let $\tilde{\iota}_+^*$ denote the adjoint relation (from $\mathcal{X}_0$ to $\mathcal{X}_+$) of the embedding mapping $\tilde{\iota}_+$ in the previous lemma. Then $\tilde{\iota}_+^*$ is an operator (single-valued, i.e. $\operatorname{mul}\tilde{\iota}_+^* = \{0\}$) and $\ker\tilde{\iota}_+^* = \{0\}$. Its domain coincides with D_- and $\tilde{\iota}_+^*\iota_-\colon D_- \subseteq \mathcal{X}_- \to \mathcal{X}_+$ is isometric.*

If $\ker\overline{\tilde{\iota}_+} = \{0\}$, then $\operatorname{ran}\tilde{\iota}_+^*$ is dense in $\mathcal{X}_+$.

Proof. The density of the domain of $\tilde{\iota}_+$ yields $\operatorname{mul}\tilde{\iota}_+^* = (\operatorname{dom}\tilde{\iota}_+)^\perp = \{0\}$, and $\overline{\operatorname{ran}\tilde{\iota}_+}^{\mathcal{X}_0} = \mathcal{X}_0$ yields $\ker\tilde{\iota}_+^* = \{0\}$. The following equivalences show $\operatorname{dom}\tilde{\iota}_+^* = D_-$:

$$\begin{aligned} g \in \operatorname{dom}\tilde{\iota}_+^* &\Leftrightarrow \langle g, \tilde{\iota}_+ f\rangle_{\mathcal{X}_0} \text{ is continuous in } f \in \tilde{D}_+ \text{ w.r.t. } \|.\|_{\mathcal{X}_+} \\ &\Leftrightarrow \sup_{f\in\tilde{D}_+\setminus\{0\}} \frac{|\langle g, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} < +\infty \\ &\Leftrightarrow g \in D_-. \end{aligned}$$

For $g \in D_- \subseteq \mathcal{X}_-$ we have

$$\|g\|_{\mathcal{X}_-} = \sup_{f\in\tilde{D}_+\setminus\{0\}} \frac{|\langle \iota_- g, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} = \sup_{f\in\tilde{D}_+\setminus\{0\}} \frac{|\langle \tilde{\iota}_+^*\iota_- g, f\rangle_{\mathcal{X}_+}|}{\|f\|_{\mathcal{X}_+}} = \|\tilde{\iota}_+^*\iota_- g\|_{\mathcal{X}_+},$$

which proves that $\tilde{\iota}_+^*\iota_-$ is isometric.

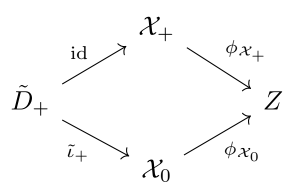
If $\ker\overline{\tilde{\iota}_+} = \{0\}$, then the following equation implies the density of $\operatorname{ran}\tilde{\iota}_+^*$ in $\mathcal{X}_+$

$$\{0\} = \ker\overline{\tilde{\iota}_+} = \ker\tilde{\iota}_+^{**} = (\operatorname{ran}\tilde{\iota}_+^*)^\perp. \qquad ❑$$

Remark 5.7. As mentioned in Remark 5.3 every $g \in D_-$ can be regarded as an element of $\mathcal{X}_+'$ by the continuous extension of $\tilde{D}_+ \ni f \mapsto \langle g, \tilde{\iota}_+ f\rangle_{\mathcal{X}_0}$ on $\mathcal{X}_+$. Since $D_- = \operatorname{dom}\tilde{\iota}_+^*$, this extension equals $\langle \tilde{\iota}_+^* g, .\rangle_{\mathcal{X}_+}$.

Proposition 5.8. *The following assertions are equivalent.*

(i) *There is a Hausdorff topological vector space $(Z, \mathcal{T})$ and two continuous embeddings $\phi_{\mathcal{X}_+}\colon \mathcal{X}_+ \to Z$ and $\phi_{\mathcal{X}_0}\colon \mathcal{X}_0 \to Z$ such that the diagram*



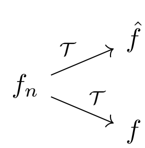
commutes.

(ii) *If $\tilde{D}_+ \ni f_n \to 0$ w.r.t. $\|.\|_{\mathcal{X}_+}$ and $\lim_{n\to\infty} f_n$ exists w.r.t. $\|.\|_{\mathcal{X}_0}$, then this limit is also 0 and if $\tilde{D}_+ \ni f_n \to 0$ w.r.t. $\|.\|_{\mathcal{X}_0}$ and $\lim_{n\to\infty} f_n$ exists w.r.t. $\|.\|_{\mathcal{X}_+}$, then this limit is also 0.*

(iii) *$\tilde{\iota}_+\colon \tilde{D}_+ \subseteq \mathcal{X}_+ \to \mathcal{X}_0, f \mapsto f$ is closable (as an operator) and its closure is injective.*

(iv) *D_- is dense in $\mathcal{X}_0$ and dense in $\mathcal{X}_+'$.*

Proof. (i) ⇒ (ii): Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\tilde{D}_+$ such that $f_n \to \hat{f}$ w.r.t. $\mathcal{X}_+$ and $f_n \to f$ w.r.t. $\mathcal{X}_0$. Since $\mathcal{T}$ is coarser than both of the topologies induced by these norms, we also have



in Z.

Since $\mathcal{T}$ is Hausdorff, we conclude $f = \hat{f}$. Hence, if either $\hat{f}$ or f is 0, then also the other is 0.

(ii) $\Rightarrow$ (iii): If $(f_n, f_n)_{n\in\mathbb{N}}$ is a sequence in $\tilde{\iota}_+$ that converges to $(0, f) \in \mathcal{X}_+ \times \mathcal{X}_0$, then $f = 0$ by (ii). Hence, $\operatorname{mul}\overline{\tilde{\iota}_+} = \{0\}$ and consequently $\tilde{\iota}_+$ is closable. On the other hand, if $(f_n, f_n)_{n\in\mathbb{N}}$ is a sequence in $\tilde{\iota}_+$ that converges to $(f, 0)$, then $f = 0$ by (ii). Consequently, $\ker\overline{\tilde{\iota}_+} = \{0\}$ and $\overline{\tilde{\iota}_+}$ is injective.

(iii) $\Rightarrow$ (iv): We have $(\operatorname{dom}\tilde{\iota}_+^*)^\perp = \operatorname{mul}\tilde{\iota}_+^{**} = \operatorname{mul}\overline{\tilde{\iota}_+}$. Since $\tilde{\iota}_+$ is closable, we have $\operatorname{mul}\overline{\tilde{\iota}_+} = \{0\}$, which yields that $\operatorname{dom}\tilde{\iota}_+^*$ is dense in $\mathcal{X}_0$. By Lemma 5.6 $\operatorname{dom}\tilde{\iota}_+^*$ coincides with D_-.

The second assertion of Lemma 5.6 yields that $\operatorname{ran}\tilde{\iota}_+^* = \tilde{\iota}_+^* D_-$ is dense in $\mathcal{X}_+$. As mentioned in Remark 5.7 every element $g \in D_-$ can be identified with $\langle \tilde{\iota}_+^* g, .\rangle_{\mathcal{X}_+} \in \mathcal{X}_+'$. Therefore, the density of $\tilde{\iota}_+^* D_-$ in $\mathcal{X}_+$ implies the density of D_- in $\mathcal{X}_+'$, because $f \mapsto \langle f, .\rangle_{\mathcal{X}_+}$ is a unitary mapping between $\mathcal{X}_+$ and $\mathcal{X}_+'$.

(iv) $\Rightarrow$ (i): Let $Y \coloneqq D_-$ be equipped with $\|g\|_Y \coloneqq \|g\|_{\mathcal{X}_-\cap\mathcal{X}_0} = \sqrt{\|g\|_{\mathcal{X}_-}^2 + \|g\|_{\mathcal{X}_0}^2}$. We define $Z \coloneqq Y'$ as the (anti)dual of Y. Then we have

$$\begin{aligned} &|\langle f, g\rangle_{\mathcal{X}_0}| \leq \|f\|_{\mathcal{X}_0}\|g\|_{\mathcal{X}_0} \leq \|f\|_{\mathcal{X}_0}\|g\|_Y && \text{for} \quad f \in \mathcal{X}_0, g \in Y \\ \text{and} \quad &|\langle f, \tilde{\iota}_+^* g\rangle_{\mathcal{X}_+}| \leq \|f\|_{\mathcal{X}_+}\underbrace{\|\tilde{\iota}_+^* g\|_{\mathcal{X}_+}}_{=\|g\|_{\mathcal{X}_-}} \leq \|f\|_{\mathcal{X}_+}\|g\|_Y && \text{for} \quad f \in \mathcal{X}_+, g \in Y. \end{aligned}$$

Hence, $\phi_{\mathcal{X}_0} \colon f \mapsto \langle f, .\rangle_{\mathcal{X}_0}$ and $\phi_{\mathcal{X}_+} \colon f \mapsto \langle f, \tilde{\iota}_+^* .\rangle_{\mathcal{X}_+}$ are continuous mappings from $\mathcal{X}_0$ and $\mathcal{X}_+$, respectively, into Z. The injectivity of these mappings follows from the density of D_- in $\mathcal{X}_0$ and D_- in $\mathcal{X}_+'$ ($\tilde{\iota}_+^* D_-$ dense in $\mathcal{X}_+$), respectively. For $f \in \tilde{D}_+$ we have

$$\phi_{\mathcal{X}_+} f = \langle f, \tilde{\iota}_+^* .\rangle_{\mathcal{X}_+} = \langle \tilde{\iota}_+ f, .\rangle_{\mathcal{X}_0} = \phi_{\mathcal{X}_0} \circ \tilde{\iota}_+ f$$

and consequently the diagram in (i) commutes. ❑

If one and therefore all assertions in Proposition 5.8 are satisfied, then $\mathcal{X}_+ \cap \mathcal{X}_0$ is defined as the intersection in Z and complete with the norm $\|.\|_{\mathcal{X}_+\cap\mathcal{X}_0} \coloneqq \sqrt{\|.\|_{\mathcal{X}_+}^2 + \|.\|_{\mathcal{X}_0}^2}$. Moreover, we define D_+ as the closure of $\tilde{D}_+$ in $\mathcal{X}_+ \cap \mathcal{X}_0$ (w.r.t. $\|.\|_{\mathcal{X}_+\cap\mathcal{X}_0}$). Note that although $\mathcal{X}_+ \cap \mathcal{X}_0$ may depend on Z, D_+ is independent of Z. We will denote the extension of $\tilde{\iota}_+$ to D_+ by ι_+, which can be expressed by $\iota_+ = \overline{\tilde{\iota}_+}$. The adjoint ι_+^* coincides with $\tilde{\iota}_+^*$. Also D_- does not change, if we replace $\tilde{D}_+$ by D_+ in Definition 5.2 and all previous results in this section also hold for D_+ and ι_+ instead of $\tilde{D}_+$ and $\tilde{\iota}_+$, respectively. If $\tilde{\iota}_+$ is already closed, then $D_+ = \tilde{D}_+$.

Lemma 5.9. *Let one assertion in Proposition 5.8 be satisfied. Let $Z = Y'$, where $Y = D_-$ endowed with $\|g\|_Y \coloneqq \|g\|_{\mathcal{X}_-\cap\mathcal{X}_0} = \sqrt{\|g\|_{\mathcal{X}_-}^2 + \|g\|_{\mathcal{X}_0}^2}$ (from Proposition 5.8 (iv) $\Rightarrow$ (i)). Then we have the following characterization for D_+:*

- $D_+ = \operatorname{dom}\iota_-^*$,
- $D_+ = \mathcal{X}_+ \cap \mathcal{X}_0$ *in* Y'.

Proof. Note that for $g \in D_-$ we have $g = (\iota_+^*)^{-1}\iota_+^* g$ and that $\iota_+^*\iota_-$ is isometric from $\operatorname{dom}\iota_-$ onto $\operatorname{dom}(\iota_+^*)^{-1}$. The following equivalences show the first assertion:

$$\begin{aligned} f \in \operatorname{dom}\iota_-^* &\Leftrightarrow D_- \ni g \mapsto \langle f, \iota_- g\rangle_{\mathcal{X}_0} \text{ is continious w.r.t. } \|.\|_{\mathcal{X}_-} \\ &\Leftrightarrow D_- \ni g \mapsto \langle f, (\iota_+^*)^{-1}\iota_+^*\iota_- g\rangle_{\mathcal{X}_0} \text{ is continious w.r.t. } \|.\|_{\mathcal{X}_-} \\ &\Leftrightarrow f \in \operatorname{dom}\left((\iota_+^*)^{-1}\right)^* = \operatorname{dom}\iota_+^{-1} = \operatorname{ran}\iota_+ = D_+. \end{aligned}$$

We define $P_+ := \mathcal{X}_+ \cap \mathcal{X}_0$ and we define P_- analogously to D_- in Definition 5.2:

$$\|g\|_{P_-} := \sup_{f\in P_+\setminus\{0\}} \frac{|\langle g, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} \quad \text{and} \quad P_- := \{g \in \mathcal{X}_0 : \|g\|_{P_-} < +\infty\}.$$

Clearly, $\|g\|_{\mathcal{X}_-} \leq \|g\|_{P_-}$ for $g \in P_-$ and consequently $P_- \subseteq D_-$. Furthermore, we can define $\iota_{P_+}\colon P_+ \subseteq \mathcal{X}_+ \to \mathcal{X}_0, f \mapsto f$ analogously to $\tilde{\iota}_+$. Note that ι_{P_+} is closed due the completeness of $(\mathcal{X}_+ \cap \mathcal{X}_0, \|.\|_{\mathcal{X}_+\cap\mathcal{X}_0})$. Then we have $\operatorname{dom}\iota_{P_+}^* = P_-$ and $\tilde{\iota}_+ \subseteq \iota_{P_+}$ and therefore $\iota_{P_+}^* \subseteq \tilde{\iota}_+^*$. For $g \in D_-$ and $f \in P_+$ we have, by definition of $P_+ = \mathcal{X}_+ \cap \mathcal{X}_0$ in Z,

$$|\langle g, f\rangle_{\mathcal{X}_0}| = |\langle \tilde{\iota}_+ g, f\rangle_{\mathcal{X}_+}| \leq \|\tilde{\iota}_+ g\|_{\mathcal{X}_+}\|f\|_{\mathcal{X}_+} = \|g\|_{\mathcal{X}_-}\|f\|_{\mathcal{X}_+},$$

which yields $\|g\|_{P_-} \leq \|g\|_{\mathcal{X}_-}$. Hence, $P_- = D_-$, $\iota_{P_+}^* = \tilde{\iota}_+^*$ and $\iota_{P_+} = \overline{\tilde{\iota}_+}$, which is equivalent to $P_+ = \mathcal{X}_+ \cap \mathcal{X}_0 = \overline{\tilde{D}_+}^{\mathcal{X}_+\cap\mathcal{X}_0} = D_+$. ❑

Theorem 5.10. *Let one assertion in Proposition 5.8 be satisfied. Then the mapping $\iota_+^*\iota_-$ can be uniquely extended to a isometric and surjective operator $\Psi\colon \mathcal{X}_- \to \mathcal{X}_+$. In particular $\mathcal{X}_-$ is a Hilbert space whose (original) norm is induced by $\langle g, f\rangle_{\mathcal{X}_-} := \langle \Psi g, \Psi f\rangle_{\mathcal{X}_+}$ and Ψ is unitary.*

Proof. By Lemma 5.6 $\iota_+^*\iota_-\colon D_- \subseteq \mathcal{X}_- \to \mathcal{X}_+$ is an isometry with dense range, since ι_+ is closed and injective by assumption. Since D_- is dense in $\mathcal{X}_-$ by construction, we can extend $\iota_+^*\iota_-$ by continuity to $\mathcal{X}_-$. We denote this extension by Ψ. For an arbitrary $g \in \mathcal{X}_-$ there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in D_- that converges to g (w.r.t. $\|.\|_{\mathcal{X}_-}$). Hence,

$$\|\Psi g\|_{\mathcal{X}_+} = \lim_{n\to\infty}\|\Psi g_n\|_{\mathcal{X}_+} = \lim_{n\to\infty}\|\iota_+^*\iota_- g_n\|_{\mathcal{X}_+} = \lim_{n\to\infty}\|g_n\|_{\mathcal{X}_-} = \|g\|_{\mathcal{X}_-}.$$

This yields that Ψ is isometric and $\operatorname{ran}\Psi$ is closed in $\mathcal{X}_+$. Since $\operatorname{ran}\Psi$ also contains the dense subspace $\operatorname{ran}\iota_+^*$, the mapping Ψ is surjective.

Clearly, this implies that $\|.\|_{\mathcal{X}_-}$ is induced by the inner product $\langle .,.\rangle_{\mathcal{X}_-} = \langle \Psi ., \Psi .\rangle_{\mathcal{X}_+}$ and $\mathcal{X}_-$ Hilbert space endowed with this inner product. Moreover, Ψ is then unitary. ❑

Corollary 5.11. *If one assertion in Proposition 5.8 is satisfied, then $\mathcal{X}_-$ can be identified with the (anti)dual space of $\mathcal{X}_+$ by*

$$\Lambda\colon \begin{cases} \mathcal{X}_- & \to \quad \mathcal{X}_+', \\ \quad g & \mapsto \quad \langle \Psi g, .\rangle_{\mathcal{X}_+}, \end{cases}$$

where Ψ is the mapping from Theorem 5.10.

If $\mathcal{X}_1$ and $\mathcal{X}_2$ are Hilbert spaces and $\mathcal{X}_2$ can be identified with the dual space of $\mathcal{X}_1$ by a unitary mapping $\Lambda : \mathcal{X}_2 \to \mathcal{X}_1'$, then we define

$$\langle g, f\rangle_{\mathcal{X}_2,\mathcal{X}_1} := \langle \Lambda g, f\rangle_{\mathcal{X}_1',\mathcal{X}_1} = (\Lambda g)(f).$$

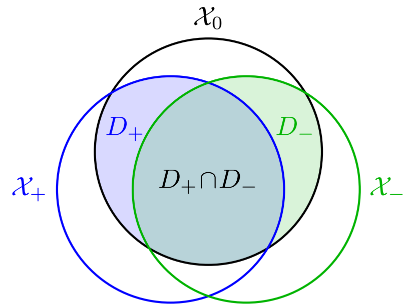


FIGURE 1. Illustration of a quasi Gelfand triple, where $D_+ = \operatorname{dom} \iota_+$ and $D_- = \operatorname{dom} \iota_-$.

Remark 5.12. For $f \in D_+$ and $g \in D_-$ we have

$$\langle g, f \rangle_{\mathcal{X}_-,\mathcal{X}_+} = \langle \Psi g, f \rangle_{\mathcal{X}_+} = \langle \iota_+^* \iota_- g, f \rangle_{\mathcal{X}_+} = \langle \iota_- g, \iota_+ f \rangle_{\mathcal{X}_0} = \langle g, f \rangle_{\mathcal{X}_0}.$$

Since these two sets are dense in $\mathcal{X}_+$ and $\mathcal{X}_-$ respectively, we have for $f \in \mathcal{X}_+$ and $g \in \mathcal{X}_-$

$$\langle g, f \rangle_{\mathcal{X}_-,\mathcal{X}_+} = \lim_{(n,m)\to(\infty,\infty)} \langle g_n, f_m \rangle_{\mathcal{X}_0},$$

where $(f_m)_{m\in\mathbb{N}}$ is a sequence in D_+ that converges to f in $\mathcal{X}_+$ and $(g_n)_{n\in\mathbb{N}}$ is a sequence in D_- that converges to g in $\mathcal{X}_-$.

Definition 5.13. Let $\mathcal{X}_+$, $\mathcal{X}_0$ and $\mathcal{X}_-$ be Hilbert spaces, where $\mathcal{X}_-$ can be identified with $\mathcal{X}_+'$. Furthermore, let $\iota_+ \colon \operatorname{dom} \iota_+ \subseteq \mathcal{X}_+ \to \mathcal{X}_0$ and $\iota_- \colon \operatorname{dom} \iota_- \subseteq \mathcal{X}_- \to \mathcal{X}_0$ be densely defined, closed, and injective linear mappings with dense range. We call $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ a *quasi Gelfand triple*, if

$$\langle g, f \rangle_{\mathcal{X}_-,\mathcal{X}_+} = \langle \iota_- g, \iota_+ f \rangle_{\mathcal{X}_0} \tag{5.1}$$

for all $f \in \operatorname{dom} \iota_+$ and $g \in \operatorname{dom} \iota_-$, and $\operatorname{dom} \iota_+^* = \operatorname{ran} \iota_-$. The space $\mathcal{X}_0$ will be referred as *pivot space*. We define $\mathcal{X}_+ \cap \mathcal{X}_0 \coloneqq \operatorname{ran} \iota_+$ and $\mathcal{X}_- \cap \mathcal{X}_0 \coloneqq \operatorname{ran} \iota_-$.

Figure 1 illustrates the setting of a quasi Gelfand triple. Since $\mathcal{X}_-$ can be identified with $\mathcal{X}_+'$ and $\mathcal{X}_+'$ can be identified with $\mathcal{X}_+$, there exists a unitary operator $\Psi \colon \mathcal{X}_- \to \mathcal{X}_+$. In fact, by (5.1) this Ψ is the extension of $\iota_+^* \iota_-$, which already appeared in Theorem 5.10. We will show this in detail in Proposition 5.16. We will call Ψ the *duality map* of the quasi Gelfand triple.

In contrast to "ordinary" Gelfand triple, the setting for quasi Gelfand triple is somehow "symmetric", i.e. the roles of $\mathcal{X}_+$ and $\mathcal{X}_-$ are interchangeable, since neither ι_+ nor ι_- have to be continuous, as indicated in the beginning of this section.

Lemma 5.14. *Let $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ with ι_+ and ι_- satisfy all conditions of Definition 5.13 except* $\operatorname{dom} \iota_+^* = \operatorname{ran} \iota_-$. *Then*

$$\operatorname{dom} \iota_+^* = \operatorname{ran} \iota_- \quad \Leftrightarrow \quad \operatorname{dom} \iota_-^* = \operatorname{ran} \iota_+.$$

In particular, if $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a quasi Gelfand triple, then also $\operatorname{dom} \iota_-^* = \operatorname{ran} \iota_+$ *holds true.*

The proof of this is basically the first part of the proof of Lemma 5.9.

Proof. By (5.1), it is clear that $\operatorname{dom}\iota_+^* \supseteq \operatorname{ran}\iota_-$ and $\operatorname{dom}\iota_-^* \supseteq \operatorname{ran}\iota_+$ holds. Moreover, for $f \in \operatorname{dom}\iota_+$, $g \in \operatorname{dom}\iota_-$ and the duality mapping Ψ we have

$$\langle g, \Psi^* f\rangle_{\mathcal{X}_-} = \langle \Psi g, f\rangle_{\mathcal{X}_+} = \langle g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} = \langle \iota_- g, \iota_+ f\rangle_{\mathcal{X}_0},$$

which implies $\iota_+^*\iota_- \subseteq \Psi$ and $\iota_-^*\iota_+ \subseteq \Psi^*$. In particular, both $\iota_+^*\iota_-$ and $\iota_-^*\iota_+$ are isometric.

Let $\operatorname{dom}\iota_+^* = \operatorname{ran}\iota_-$. Then $\iota_+^*\iota_-$ is isometric from $\operatorname{dom}\iota_-$ onto $\operatorname{dom}(\iota_+^*)^{-1}$. The following equivalences

$$\begin{aligned} f \in \operatorname{dom}\iota_-^* &\Leftrightarrow \operatorname{dom}\iota_- \ni g \mapsto \langle f, \iota_- g\rangle_{\mathcal{X}_0} \text{ is continuous w.r.t. } \|.\|_{\mathcal{X}_-} \\ &\Leftrightarrow \operatorname{dom}\iota_- \ni g \mapsto \langle f, (\iota_+^*)^{-1}\iota_+^*\iota_- g\rangle_{\mathcal{X}_0} \text{ is continuous w.r.t. } \|.\|_{\mathcal{X}_-} \\ &\Leftrightarrow f \in \operatorname{dom}\left((\iota_+^*)^{-1}\right)^* = \operatorname{dom}\iota_+^{-1} = \operatorname{ran}\iota_+ \end{aligned}$$

imply $\operatorname{dom}\iota_-^* = \operatorname{ran}\iota_+$.

The other implication follows analogously. ❑

Lemma 5.15. *Let $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ with ι_+ and ι_- satisfy all conditions of Definition 5.13 except $\operatorname{dom}\iota_+^* = \operatorname{ran}\iota_-$. Then there exists an extension $\hat\iota_-$ of ι_- that respects (5.1) and satisfies $\operatorname{dom}\iota_+^* = \operatorname{ran}\hat\iota_-$. In particular, $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ with ι_+ and $\hat\iota_-$ forms a quasi Gelfand triple.*

Proof. Note that

$$g \in \operatorname{dom}\iota_+^* \Leftrightarrow \operatorname{dom}\iota_+ \ni f \mapsto \langle g, \iota_+ f\rangle_{\mathcal{X}_0} \text{ is continuous w.r.t. } \|.\|_{\mathcal{X}_+}.$$

Hence, for $g \in \operatorname{dom}\iota_+^*$ there exists an $h \in \mathcal{X}_-$ such that

$$\langle g, \iota_+ f\rangle_{\mathcal{X}_0} = \langle h, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} \quad \text{for all} \quad f \in \operatorname{dom}\iota_+. \tag{5.2}$$

We define $\phi(g) := h$ for $g \in \operatorname{dom}\iota_+$. Clearly, $\phi(g) = \iota_-^{-1} g$ for $g \in \operatorname{ran}\iota_-$. Therefore, $\hat\iota_- := \phi^{-1}$ is an extension of ι_- that satisfies $\operatorname{dom}\iota_+^* = \operatorname{ran}\hat\iota_-$. Moreover, by (5.2) we have $\langle \hat\iota_- g, \iota_+ f\rangle_{\mathcal{X}_0} = \langle g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+}$ for $g \in \operatorname{dom}\hat\iota_-$ and $f \in \operatorname{dom}\iota_+$. ❑

Proposition 5.16. *Let $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ be a quasi Gelfand triple and $\Psi\colon \mathcal{X}_- \to \mathcal{X}_+$ be its duality map. Then*

$$\overline{\iota_+^*\iota_-} = \Psi, \quad \overline{\iota_-^*\iota_+} = \Psi^*, \quad \iota_+^* = \Psi\iota_-^{-1} \quad \textit{and} \quad \iota_-^* = \Psi^*\iota_+^{-1}.$$

Proof. Let $f \in \operatorname{dom}\iota_+$ and $g \in \operatorname{dom}\iota_-$. Then

$$\langle \Psi g, f\rangle_{\mathcal{X}_+} = \langle g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} = \langle \iota_- g, \iota_+ f\rangle_{\mathcal{X}_0} = \langle \iota_+^*\iota_- g, f\rangle_{\mathcal{X}_+}.$$

Since $\operatorname{dom}\iota_+$ is dense in $\mathcal{X}_+$, we have $\Psi g = \iota_+^*\iota_- g$ for all $g \in \operatorname{dom}\iota_-$. Applying ι_-^{-1} on both sides gives $\Psi\iota_-^{-1} = \iota_+^*$. Moreover, the density of $\operatorname{dom}\iota_-$ in $\mathcal{X}_-$ yields $\Psi = \overline{\iota_+^*\iota_-}$.

Analogously, we can show $\Psi^*\iota_+^{-1} = \iota_-^*$ and $\Psi^* = \overline{\iota_-^*\iota_+}$. ❑

Theorem 5.17. *Let $\mathcal{X}_+$ and $\mathcal{X}_0$ be Hilbert spaces and $\iota_+\colon \operatorname{dom}\iota_+ \subseteq \mathcal{X}_+ \to \mathcal{X}_0$ be a densely defined, closed, and injective linear mapping with dense range. Then there exists a Hilbert space $\mathcal{X}_-$ and a mapping ι_- such that $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a quasi Gelfand triple.*

In particular, $\mathcal{X}_-$ is given by Definition 5.2, where $D_+ = \operatorname{ran}\iota_+$.

Proof. We will identify $\operatorname{dom}\iota_+$ with $\operatorname{ran}\iota_+$ and denote it by D_+. Then item (iii) of Proposition 5.8 is satisfied. Hence, the corresponding D_- (Definition 5.2) is dense in $\mathcal{X}_0$ and its completion $\mathcal{X}_-$ (w.r.t. to $\|.\|_{\mathcal{X}_-}$) can be identified with $\mathcal{X}_+'$ by Corollary 5.11. The linear mapping

$$\iota_- : \begin{cases} D_- \subseteq \mathcal{X}_- & \to & \mathcal{X}_0, \\ g & \mapsto & g, \end{cases}$$

is densely defined and injective by construction. By the already shown $\operatorname{ran} \iota_- = D_-$ is dense in $\mathcal{X}_0$. Finally, by Lemma 5.5 ι_- is closed and by Lemma 5.6 $\operatorname{dom} \iota_+^* = D_- = \operatorname{ran} \iota_-$. ❑

Remark 5.18. By Theorem 5.17 the setting in the beginning of the section establishes a quasi Gelfand triple, if one assertion of Proposition 5.8 is satisfied.

Until the end of this section we will assume that $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a quasi Gelfand triple and we will identify $\operatorname{dom} \iota_+$ with $\operatorname{ran} \iota_+$ and denote it by D_+. The set D_- is defined by Definition 5.2 for D_+. This set coincides with $\operatorname{ran} \iota_-$, which we will identify with $\operatorname{dom} \iota_-$.

Proposition 5.19. *The space $D_+ \cap D_-$ is complete with respect to $\|.\|_{\mathcal{X}_+ \cap \mathcal{X}_-} \coloneqq \sqrt{\|.\|_{\mathcal{X}_+}^2 + \|.\|_{\mathcal{X}_-}^2}$.*

Proof. For $f \in D_+ \cap D_-$ we have

$$\|f\|_{\mathcal{X}_0}^2 = |\langle f, f\rangle_{\mathcal{X}_0}| = |\langle f, f\rangle_{\mathcal{X}_-,\mathcal{X}_+}| \leq \|f\|_{\mathcal{X}_-} \|f\|_{\mathcal{X}_+} \leq \|f\|_{\mathcal{X}_+ \cap \mathcal{X}_-}^2.$$

Hence, every Cauchy sequence in $D_+ \cap D_-$ with respect to $\|.\|_{\mathcal{X}_+ \cap \mathcal{X}_-}$ is also a Cauchy sequence with respect to $\|.\|_{\mathcal{X}_0}$, $\|.\|_{\mathcal{X}_+}$ and $\|.\|_{\mathcal{X}_-}$.

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $D_+ \cap D_-$ with respect to $\|.\|_{\mathcal{X}_+ \cap \mathcal{X}_-}$. By the closedness of ι_+ the limit with respect to $\|.\|_{\mathcal{X}_0}$ and the limit with respect to $\|.\|_{\mathcal{X}_+}$ coincide. The same argument for ι_- yields that the limit with respect to $\|.\|_{\mathcal{X}_0}$ and the limit with respect $\|.\|_{\mathcal{X}_-}$ also coincide. Therefore, all these limits have to coincide and $(f_n)_{n\in\mathbb{N}}$ converges to that limit in $D_+ \cap D_-$ w.r.t. $\|.\|_{\mathcal{X}_+ \cap \mathcal{X}_-}$. ❑

Lemma 5.20. *The operator*

$$\begin{bmatrix} \iota_+ & \iota_- \end{bmatrix} : \begin{cases} D_+ \times D_- \subseteq \mathcal{X}_+ \times \mathcal{X}_- & \to & \mathcal{X}_0, \\ \begin{bmatrix} f \\ g \end{bmatrix} & \mapsto & f + g, \end{cases}$$

is closed.

Proof. Let $\left(\left(\left[\begin{smallmatrix} f_n \\ g_n \end{smallmatrix}\right], z_n\right)\right)_{n\in\mathbb{N}}$ be a sequence in $\begin{bmatrix} \iota_+ & \iota_- \end{bmatrix}$ that converges to $\left(\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right], z\right)$ in $\mathcal{X}_+ \times \mathcal{X}_- \times \mathcal{X}_0$. Then we have

$$\|z\|_{\mathcal{X}_0}^2 = \lim_{n\to\infty} \|f_n + g_n\|_{\mathcal{X}_0}^2 = \lim_{n\to\infty} \left(\|f_n\|_{\mathcal{X}_0}^2 + \|g_n\|_{\mathcal{X}_0}^2 + 2\operatorname{Re}\langle f_n, g_n\rangle_{\mathcal{X}_0}\right).$$

Since $2\operatorname{Re}\langle f_n, g_n\rangle_{\mathcal{X}_0}$ converges to $2\operatorname{Re}\langle f, g\rangle_{\mathcal{X}_+,\mathcal{X}_-}$, we conclude that $\|f_n\|_{\mathcal{X}_0}$ and $\|g_n\|_{\mathcal{X}_0}$ are bounded. Hence, there exists a subsequence of $(f_n)_{n\in\mathbb{N}}$ that converges weakly to an $\tilde{f} \in \mathcal{X}_0$. Moreover, by Lemma A.3 we can pass on to a further subsequence $(f_{n(k)})_{k\in\mathbb{N}}$ such that $\left(\frac{1}{j}\sum_{k=1}^{j} f_{n(k)}\right)_{j\in\mathbb{N}}$ converges to $\tilde{f}$ strongly (w.r.t. $\|.\|_{\mathcal{X}_0}$). The sequence $\left(\frac{1}{j}\sum_{k=1}^{j} f_{n(k)}\right)_{j\in\mathbb{N}}$ has still the limit f in $\mathcal{X}_+$ (w.r.t. $\|.\|_{\mathcal{X}_+}$) and because ι_+ is closed we conclude that $f = \tilde{f} \in D_+$. By linearity we also have $\frac{1}{j}\sum_{k=1}^{j} g_{n(k)} \to z - f$ in $\mathcal{X}_0$ for the same subsequence. Since $\frac{1}{j}\sum_{k=1}^{j} g_{n(k)}$ is a Cauchy sequence in both $\mathcal{X}_-$ and $\mathcal{X}_0$, the closedness of ι_- gives that $g = z - f \in D_-$. Hence, $z = \begin{bmatrix} \iota_+ & \iota_- \end{bmatrix} \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ and the operator $\begin{bmatrix} \iota_+ & \iota_- \end{bmatrix}$ is closed. ❑

Proposition 5.21. *$D_+ \cap D_-$ is dense in $\mathcal{X}_0$ with respect to $\|.\|_{\mathcal{X}_0}$.*

Proof. By $\operatorname{dom} \iota_\pm^* = D_\mp$ (Lemma 5.14) we have

$$\mathcal{X}_0 = \left(\operatorname{mul}\begin{bmatrix}\iota_+ & \iota_-\end{bmatrix}\right)^\perp = \overline{\operatorname{dom}\begin{bmatrix}\iota_+ & \iota_-\end{bmatrix}^*} = \overline{\operatorname{dom}\iota_+^* \cap \operatorname{dom}\iota_-^*} = \overline{D_- \cap D_+}. \qquad ❑$$

The following theorem can be found in [17, Theorem 2 p. 200], we just changed that the operator maps into a different space, which does not change the proof.

Theorem 5.22 (J. von Neumann)**.** *Let T be a closed linear operator from the Hilbert spaces X to the Hilbert space Y. Then T^*T and TT^* are self-adjoint, and $(\mathrm{I}_X + T^*T)$ and $(\mathrm{I}_Y + TT^*)$ are boundedly invertible.*

Corollary 5.23. *The set $D_+ \cap D_-$ is dense in $\mathcal{X}_+$ and $\mathcal{X}_-$ with respect to their corresponding norms.*

Proof. Applying Theorem 5.22 to ι_+ yields $\iota_+^*\iota_+$ is self-adjoint. Hence, $\operatorname{dom}\iota_+^*\iota_+$ is dense in $\mathcal{X}_+$. By Lemma 5.14 $\operatorname{dom}\iota_+^* = D_-$, consequently $\operatorname{dom}\iota_+^*\iota_+ = D_+ \cap D_-$.

An analogous argument for ι_- yields $D_+ \cap D_-$ is dense in $\mathcal{X}_-$. ❑

Corollary 5.24. *$D_+ + D_- = \mathcal{X}_0$.*

Proof. Applying Theorem 5.22 to ι_+ gives that $(\mathrm{I}_{\mathcal{X}_0} + \iota_+\iota_+^*)$ is onto. Hence, for every $x \in \mathcal{X}_0$ there exists a $g_x \in \operatorname{dom}\iota_+\iota_+^* \subseteq D_-$ such that

$$x = \underbrace{g_x}_{\in D_-} + \underbrace{\iota_+\iota_+^* g_x}_{\in D_+}.$$

Since $g_x \in \operatorname{dom}\iota_+\iota_+^*$, we have $\iota_+^* g_x \in D_+$ and consequently $x \in D_+ + D_-$. ❑

Proposition 5.25. *Let T be a bounded and boundedly invertible mapping from $\mathcal{X}_0$ to another Hilbert space $\mathcal{Y}_0$. Then $P_+ \coloneqq TD_+$ equipped with $\|f\|_{\mathcal{Y}_+} \coloneqq \|T^{-1}f\|_{\mathcal{X}_+}$ establishes a quasi Gelfand triple $(\mathcal{Y}_+, \mathcal{Y}_0, \mathcal{Y}_-)$, where $\mathcal{Y}_+$ is the completion of P_+ and $\mathcal{Y}_-$ is the completion of P_- defined as in Definition 5.2, where D_+ is replaced by P_+. Moreover, $P_- = (T^*)^{-1}D_-$ and $\|g\|_{\mathcal{Y}_-} = \|T^*g\|_{\mathcal{X}_-}$ for $g \in P_-$.*

Proof. The mapping $T\big|_{D_+} : D_+ \to P_+$ is isometric and surjective, if we equip its domain with $\|.\|_{\mathcal{X}_+}$ and its codomain with $\|.\|_{\mathcal{Y}_+}$. So the linear (single-valued) relation $\left[\begin{smallmatrix} T & 0 \\ 0 & T\end{smallmatrix}\right]\iota_+ = \{(Tf, Tg) : (f,g) \in \iota_+\} \subseteq \mathcal{Y}_+ \times \mathcal{Y}_0$ is closed. Since this linear relation coincides with the embedding $\iota_{P_+} : P_+ \subseteq \mathcal{Y}_+ \to \mathcal{Y}_0, f \mapsto f$, Theorem 5.17 yields that $(\mathcal{Y}_+, \mathcal{Y}_0, \mathcal{Y}_-)$ is a quasi Gelfand triple.

For $g \in P_-$ we have

$$\begin{aligned}\|g\|_{\mathcal{Y}_-} = \sup_{h \in P_+\setminus\{0\}} \frac{|\langle g, h\rangle_{\mathcal{Y}_0}|}{\|h\|_{\mathcal{Y}_+}} &= \sup_{f \in D_+\setminus\{0\}} \frac{|\langle g, Tf\rangle_{\mathcal{Y}_0}|}{\|Tf\|_{\mathcal{Y}_+}} \\ &= \sup_{f \in D_+\setminus\{0\}} \frac{|\langle T^*g, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} = \|T^*g\|_{\mathcal{X}_-}\end{aligned}$$

and consequently $P_- = (T^*)^{-1}D_-$. ❑

Corollary 5.26. *With the assumption from Proposition 5.25 the operators $T\big|_{D_+}$ and $(T^*)^{-1}\big|_{D_-}$ can be continuously extended to unitary operators from $\mathcal{X}_+$ and $\mathcal{X}_-$ to $\mathcal{Y}_+$ and $\mathcal{Y}_-$ respectively. These extension will be denoted by T_+ and $(T^*)_-^{-1}$. Moreover, $\langle (T^*)_-^{-1} g, T_+ f\rangle_{\mathcal{Y}_-,\mathcal{Y}_+} = \langle g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+}$ for $g \in \mathcal{X}_-$ and $f \in \mathcal{X}_+$.*

Corollary 5.27. *Let S, T be a bounded and boundedly invertible mappings on $\mathcal{X}_0$. Then* $\begin{bmatrix} ST\big|_{D_+} & S(T^*)^{-1}\big|_{D_-} \end{bmatrix}$ *is a densely defined closed surjective linear operator from $\mathcal{X}_+ \times \mathcal{X}_-$ to $\mathcal{X}_0$. In particular* $\operatorname{ran} \begin{bmatrix} ST\big|_{D_+} & S(T^*)^{-1}\big|_{D_-} \end{bmatrix} = \mathcal{X}_0$.

Proof. Let $P_+ = TD_+$. Then by Proposition 5.25 the corresponding P_- can be obtained by $(T^*)^{-1}D_-$. The mapping

$$\Xi\colon \begin{cases} \mathcal{X}_+ \times \mathcal{X}_- \times \mathcal{X}_0 & \to \quad \mathcal{Y}_+ \times \mathcal{Y}_- \times \mathcal{X}_0, \\ \begin{bmatrix} f \\ g \\ z \end{bmatrix} & \mapsto \quad \begin{bmatrix} T_+ & 0 & 0 \\ 0 & (T^*)_-^{-1} & 0 \\ 0 & 0 & S^{-1} \end{bmatrix} \begin{bmatrix} f \\ g \\ z \end{bmatrix} \end{cases}$$

is linear bounded and boundedly invertible, where $\mathcal{Y}_\pm$ is the completion of $P_\pm$ as in Proposition 5.25. Since $(\mathcal{Y}_+, \mathcal{X}_0, \mathcal{Y}_-)$ is a quasi Gelfand triple,

$$\begin{bmatrix} \iota_{P_+} & \iota_{P_-} \end{bmatrix} = \left\{ \begin{bmatrix} Tf \\ (T^*)^{-1}g \\ Tf + (T^*)^{-1}g \end{bmatrix} : f \in D_+, g \in D_- \right\}$$

is closed in $\mathcal{Y}_+ \times \mathcal{Y}_- \times \mathcal{X}_0$ (Lemma 5.20) and therefore also its pre-image under Ξ

$$\Xi^{-1}\left(\begin{bmatrix} \iota_{P_+} & \iota_{P_-} \end{bmatrix}\right) = \begin{bmatrix} T^{-1} & 0 & 0 \\ 0 & T^* & 0 \\ 0 & 0 & S \end{bmatrix} \begin{bmatrix} \iota_{P_+} & \iota_{P_-} \end{bmatrix} = \begin{bmatrix} ST\iota_+ & S(T^*)^{-1}\iota_- \end{bmatrix}$$

is closed in $\mathcal{X}_+ \times \mathcal{X}_- \times \mathcal{X}_0$. Furthermore, by Corollary 5.24

$$\operatorname{ran} \begin{bmatrix} ST\big|_{D_+} & S(T^*)^{-1}\big|_{D_-} \end{bmatrix} = S \operatorname{ran} \begin{bmatrix} \iota_{P_+} & \iota_{P_-} \end{bmatrix} = S\mathcal{X}_0 = \mathcal{X}_0. \qquad \square$$

Lemma 5.28. *Let A_0 be a densely defined, closed, skew-symmetric operator on $\mathcal{X}_0$, $\mathcal{Y}_0$ be a Hilbert space, and let $T : \mathcal{X}_0 \to \mathcal{Y}_0$ be a bounded and boundedly invertible. Let $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ be a quasi Gelfand triple such that $(\mathcal{X}_+, B_1, \Psi B_2)$ is a boundary triple for A_0^*. Furthermore, let $\mathcal{Y}_+$ and $\mathcal{Y}_-$ be as defined in Proposition 5.25. Then $(\mathcal{Y}_+, \mathcal{Y}_0, \mathcal{Y}_-)$ is also a quasi Gelfand triple such that $(\mathcal{Y}_+, T_+B_1, \Phi(T^*)_-^{-1}B_2)$ is a boundary triple for A_0^*, where Φ denotes the duality map of $(\mathcal{Y}_+, \mathcal{Y}_0, \mathcal{Y}_-)$.*

Proof. By Proposition 5.25 $(\mathcal{Y}_+, \mathcal{Y}_0, \mathcal{Y}_-)$ is a quasi Gelfand triple. For $x, y \in \operatorname{dom} A_0^*$ we have, by Corollary 5.26,

$$\begin{aligned} \langle B_1x, \Psi B_2y\rangle_{\mathcal{X}_+} &= \langle B_1x, B_2y\rangle_{\mathcal{X}_+,\mathcal{X}_-} = \langle T_+B_1x, (T^*)_-^{-1}B_2y\rangle_{\mathcal{Y}_+,\mathcal{Y}_-} \\ &= \langle T_+B_1x, \Phi(T^*)_-^{-1}B_2y\rangle_{\mathcal{Y}_+}. \end{aligned}$$

Since $T_+\colon \mathcal{X}_+ \to \mathcal{Y}_+$ and $(T^*)_-^{-1}\colon \mathcal{X}_- \to \mathcal{Y}_-$ are surjective, the surjectivity of $\begin{bmatrix} T_+B_1 \\ \Phi(T^*)_-^{-1}B_2 \end{bmatrix} = \begin{bmatrix} T_+ & 0 \\ 0 & \Phi(T^*)_-^{-1}\Psi^{-1} \end{bmatrix} \begin{bmatrix} B_1 \\ \Psi B_2 \end{bmatrix}$ follows from the surjectivity of $\begin{bmatrix} B_1 \\ \Psi B_2 \end{bmatrix}$. $\square$

Remark 5.29. In the setting of Lemma 5.28 the duality map Φ can be described by $\Phi = T_+\Psi\big((T^*)_-^{-1}\big)^{-1}$. Note that $\big((T^*)_-^{-1}\big)^{-1}$ can be described by the continuous extension of $T^*\big|_{P_-} : P_- \subseteq \mathcal{Y}_- \to \mathcal{X}_-$. We denote this extension by T_-^*. Hence, $\Phi = T_+\Psi T_-^*$.

6. **Boundary spaces.** In this section we will construct a suitable boundary space $\mathcal{V}_L$ (Definition 6.5), such that we can extend the integration by parts formula (Lemma 3.8). We will formulate the boundary conditions in this space in section 7. This space will provide a quasi Gelfand triple with a subspace of $L^2(\partial\Omega)$ as pivot space. In order to impose different boundary conditions on different parts of the boundary we introduce boundary operators that only act on a part of the boundary and their boundary spaces $\mathcal{V}_{L,\Gamma_1}$.

Definition 6.1. We say $(\Gamma_j)_{j=1}^k$, where $\Gamma_j \subseteq \partial\Omega$, is a *splitting with thin boundaries* of $\partial\Omega$, if

(i) $\bigcup_{j=1}^k \overline{\Gamma_j} = \partial\Omega$,
(ii) the sets Γ_j are pairwise disjoint,
(iii) the sets Γ_j are relatively open in $\partial\Omega$,
(iv) the boundaries of Γ_j have zero measure w.r.t. the surface measure of $\partial\Omega$.

For $\Gamma \subseteq \partial\Omega$ we will denote by P_Γ the orthogonal projection from $L^2(\partial\Omega)^{m_1}$ on $L^2_\pi(\Gamma) \coloneqq \overline{\operatorname{ran} \mathbb{1}_\Gamma L_\nu} \subseteq L^2(\Gamma)^{m_1}$, where $\mathbb{1}_M$ denotes the indicator function for a set M. We endow $L^2_\pi(\Gamma)$ with the inner product of $L^2(\partial\Omega)^{m_1}$. Therefore, we can adapt (3.1) to obtain

$$\langle L_\partial f, g\rangle_{L^2(\Omega)^{m_1}} + \langle f, L_\partial^{\mathsf{H}} g\rangle_{L^2(\Omega)^{m_2}} = \langle L_\nu \gamma_0 f, \underbrace{P_{\partial\Omega}\gamma_0 g}_{\pi_L g}\rangle_{L^2(\partial\Omega)^{m_1}}. \tag{6.1}$$

We define $\pi_L^\Gamma \colon H^1(\Omega)^{m_1} \to L^2_\pi(\Gamma)$ by $\pi_L^\Gamma \coloneqq P_\Gamma \gamma_0$ and $\pi_L \coloneqq \pi_L^{\partial\Omega}$. Since both P_Γ and γ_0 are continuous, the mapping π_L^Γ is also continuous. Therefore, $\ker \pi_L^\Gamma$ is closed. Note that $P_\Gamma = \mathbb{1}_\Gamma P_{\partial\Omega}$ and consequently $\pi_L^\Gamma = \mathbb{1}_\Gamma \pi_L$, and $\mathbb{1}_\Gamma L_\nu = L_\nu \mathbb{1}_\Gamma$.

Example 6.2. Let L be as in Example 3.3. Then $L_\nu f = \nu \cdot f$ and L_ν is certainly surjective. Therefore, $L^2_\pi(\partial\Omega) = L^2(\partial\Omega)$, $\pi_L = \gamma_0$ and $\pi_L^\Gamma = \mathbb{1}_\Gamma \gamma_0$. Since $L_\partial^{\mathsf{H}} = \operatorname{grad}$, we have $H(L_\partial^{\mathsf{H}}, \Omega) = H^1(\Omega)$.

Lemma 6.3. *Let $\Gamma \subseteq \partial\Omega$ be relatively open and let the boundary of Γ have zero measure (w.r.t. the surface measure of $\partial\Omega$). Then $\ker \pi_L^\Gamma$ is closed as subspace of $H^1(\Omega)^{m_1}$ endowed with the trace topology of $\|.\|_{H(L_\partial^{\mathsf{H}},\Omega)}$, i.e.*

$$\overline{\ker \pi_L^\Gamma}^{\|.\|_{H(L_\partial^{\mathsf{H}},\Omega)}} \cap H^1(\Omega)^{m_1} = \ker \pi_L^\Gamma.$$

Proof. Clearly, $\overline{\ker \pi_L^\Gamma}^{\|.\|_{H(L_\partial^{\mathsf{H}},\Omega)}} \cap H^1(\Omega)^{m_1} \supseteq \ker \pi_L^\Gamma$. So we will show the other inclusion. Note that for $\Upsilon \subseteq \partial\Omega$ we have

$$H^1_\Upsilon(\Omega)^{m_2} \coloneqq \left\{ f \in H^1(\Omega)^{m_2} : \mathbb{1}_\Upsilon \gamma_0 f = 0 \in L^2(\partial\Omega)^{m_2} \right\}.$$

Hence, $H^1_{\partial\Omega\setminus\Gamma}(\Omega)^{m_2} = H^1_{\partial\Omega\setminus\overline{\Gamma}}(\Omega)^{m_2}$, since the boundary of Γ has zero measure. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $\ker \pi_L^\Gamma$ which converges to $g \in H^1(\Omega)^{m_1}$ with respect to $\|.\|_{H(L_\partial^{\mathsf{H}},\Omega)}$. By Corollary 3.9 we have for an arbitrary $f \in H^1_{\partial\Omega\setminus\Gamma}(\Omega)^{m_2}$

$$|\langle L_\nu \gamma_0 f, \pi_L^\Gamma(g - g_n)\rangle_{L^2}| = |\langle L_\nu \gamma_0 f, \pi_L(g - g_n)\rangle_{L^2}| \le \|f\|_{H(L_\partial,\Omega)} \|g - g_n\|_{H(L_\partial^{\mathsf{H}},\Omega)}.$$

Since $\pi_L^\Gamma(g - g_n) = \pi_L^\Gamma g$ and the right-hand-side converges to 0, we can see that $\pi_L^\Gamma g \perp L_\nu \gamma_0 H^1_{\partial\Omega\setminus\Gamma}(\Omega)^{m_2}$. By [13, Th. 13.6.10, Re. 13.6.12] $\gamma_0 H^1_{\partial\Omega\setminus\Gamma}(\Omega)^{m_2}$ is dense in $L^2(\Gamma)^{m_2}$, which implies $\pi_L^\Gamma g \perp \operatorname{ran} \mathbb{1}_\Gamma L_\nu$. By definition $\pi_L^\Gamma g$ is also in $\overline{\operatorname{ran} \mathbb{1}_\Gamma L_\nu}$, which leads to $\pi_L^\Gamma g = 0$. Hence, $\ker \pi_L^\Gamma$ is closed in $H^1(\Omega)^{m_1}$ with respect to $\|.\|_{H(L_\partial^{\mathsf{H}},\Omega)}$. ❑

By the previous lemma

$$\|\phi\|_{M_\Gamma} := \inf\Big\{\|g\|_{H(L_\partial^{\mathsf{H}},\Omega)} : \pi_L^\Gamma g = \phi\Big\}$$

is a norm on $M_\Gamma := \operatorname{ran} \pi_L^\Gamma$. The next lemma will show that this norm is induced by an inner product.

Lemma 6.4. *Let $\Gamma \subseteq \partial\Omega$ be relatively open and let the boundary of Γ have zero measure (w.r.t. the surface measure of $\partial\Omega$). Then the space $(M_\Gamma, \|.\|_{M_\Gamma})$ is a pre-Hilbert space. Furthermore, its completion denoted by $(\overline{M_\Gamma}, \|.\|_{\overline{M_\Gamma}})$ is isomorphic to the Hilbert space $H(L_\partial^{\mathsf{H}},\Omega)\Big/\overline{\ker \pi_L^\Gamma}^{H(L_\partial^{\mathsf{H}},\Omega)}$. The mapping $\pi_L^\Gamma\colon H^1(\Omega)^{m_1} \to M_\Gamma$ can be continuously extended to a surjective contraction $\bar\pi_L^\Gamma\colon H(L_\partial^{\mathsf{H}},\Omega) \to \overline{M_\Gamma}$. The kernel of $\bar\pi_L^\Gamma$ satisfies $\ker \bar\pi_L^\Gamma = \overline{\ker \pi_L^\Gamma}^{H(L_\partial^{\mathsf{H}},\Omega)}$.*

Instead of $\bar\pi_L^{\partial\Omega}$ we will just write $\bar\pi_L$.

Proof. By Lemma 6.3 $\ker \pi_L^\Gamma$ is closed in $H^1(\Omega)^{m_1}$ with respect to trace topology of $\|.\|_{H(L_\partial^{\mathsf{H}},\Omega)}$, which implies that $\Big(H^1(\Omega)^{m_1}\Big/\ker \pi_L^\Gamma, \|.\|_{H(L_\partial^{\mathsf{H}},\Omega)/\ker \pi_L^\Gamma}\Big)$ is a normed space (normed space factorized by a closed subspace is again a normed space). Since

$$\big\|[g]_\sim\big\|_{H(L_\partial^{\mathsf{H}},\Omega)/\ker \pi_L^\Gamma} = \big\|\pi_L^\Gamma g\big\|_{M_\Gamma},$$

it is straight forward that $[g]_\sim \mapsto \pi_L^\Gamma g$ is an isometry from $\Big(H^1(\Omega)^{m_1}\Big/\ker \pi_L^\Gamma,$ $\|.\|_{H(L_\partial^{\mathsf{H}},\Omega)/\ker \pi_L^\Gamma}\Big)$ onto $(M_\Gamma, \|.\|_{M_\Gamma})$.

Clearly, $(M_\Gamma, \|.\|_{M_\Gamma})$ has a completion $(\overline{M_\Gamma}, \|.\|_{\overline{M_\Gamma}})$. By definition of the norm $\|.\|_{M_\Gamma}$ we have for every $g \in H^1(\Omega)^{m_1}$

$$\|\pi_L^\Gamma g\|_{\overline{M_\Gamma}} = \|\pi_L^\Gamma g\|_{M_\Gamma} \leq \|g\|_{H(L_\partial^{\mathsf{H}},\Omega)}.$$

Therefore, we can extend π_L^Γ by continuity on $H(L_\partial^{\mathsf{H}},\Omega)$. This extension is denoted by $\bar\pi_L^\Gamma$ and is a contraction by the previous equation.

Let $g \in H(L_\partial^{\mathsf{H}},\Omega)$. Then by Theorem 3.18 there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in $H^1(\Omega)^{m_1}$, which converges to g. Therefore, we have

$$\|\bar\pi_L^\Gamma g\|_{\overline{M_\Gamma}} = \lim_{n\to\infty} \|\pi_L^\Gamma g_n\|_{M_\Gamma} = \lim_{n\to\infty} \inf_{k\in\ker \pi_L^\Gamma} \|g_n + k\|_{H(L_\partial^{\mathsf{H}},\Omega)}.$$

The triangular inequality yields

$$\inf_{k\in\ker \pi_L^\Gamma} \|g+k\| - \|g_n - g\| \leq \inf_{k\in\ker \pi_L^\Gamma} \|g_n + k\| \leq \inf_{k\in\ker \pi_L^\Gamma} \|g+k\| + \|g_n - g\|.$$

Hence, we have

$$\|\bar\pi_L^\Gamma g\|_{\overline{M_\Gamma}} = \inf_{k\in\ker \pi_L^\Gamma} \|g+k\|_{H(L_\partial^{\mathsf{H}},\Omega)} = \inf_{k\in\overline{\ker \pi_L^\Gamma}} \|g+k\|_{H(L_\partial^{\mathsf{H}},\Omega)} \tag{6.2}$$

and consequently $H(L_\partial^{\mathsf{H}},\Omega)\Big/\overline{\ker \pi_L^\Gamma}$ is isomorphic to $\operatorname{ran} \bar\pi_L^\Gamma$. Since $H(L_\partial^{\mathsf{H}},\Omega)\Big/\overline{\ker \pi_L^\Gamma}$ is a Hilbert space, in particular complete, and $M_\Gamma \subseteq \operatorname{ran} \bar\pi_L^\Gamma \subseteq \overline{M_\Gamma}$, we have $\overline{M_\Gamma} = \operatorname{ran} \bar\pi_L^\Gamma$. This makes $\overline{M_\Gamma}$ also a Hilbert space and M_Γ a pre-Hilbert space.

Finally, equation (6.2) implies $\ker \bar\pi_L^\Gamma = \overline{\ker \pi_L^\Gamma}$. ❑

Now we are able to define a complete subspace of $H(L_\partial^{\mathsf{H}},\Omega)$ that is in some sense 0 at one part of the boundary and the corresponding boundary space for the other part of the boundary.

Definition 6.5. Let $\Gamma_0, \Gamma_1 \subseteq \partial\Omega$ be a splitting with thin boundaries and $\bar{\pi}_L$ the extension of π_L introduced in Lemma 6.4. Then we define

$$H_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega) \coloneqq \ker \bar{\pi}_L^{\Gamma_0} \quad \text{and} \quad \mathcal{V}_{L,\Gamma_1} \coloneqq \operatorname{ran} \bar{\pi}_L\big|_{H_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)},$$

where we endow $H_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)$ with $\|.\|_{H(L_\partial^{\mathsf{H}},\Omega)}$ and $\mathcal{V}_{L,\Gamma_1}$ with $\|.\|_{\mathcal{V}_{L,\Gamma_1}} \coloneqq \|.\|_{\overline{M_{\partial\Omega}}}$. Instead of $\mathcal{V}_{L,\partial\Omega} = \operatorname{ran} \bar{\pi}_L = \overline{M_{\partial\Omega}}$ we just write $\mathcal{V}_L$.

From now on until the end of this section we will assume that $\Gamma_0, \Gamma_1 \subseteq \partial\Omega$ is a splitting with thin boundaries. By Lemma 6.4 $\mathcal{V}_L$ is a Hilbert space.

Note that $\mathcal{V}_{L,\Gamma_1}$ and $\overline{M_{\Gamma_1}}$ are not necessarily the same space. Although, we have $\bar{\pi}_L^{\Gamma_1} g = \bar{\pi}_L g$ (in $L^2(\partial\Omega)^{m_1}$) for $g \in H^1(\Omega)^{m_1} \cap H_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)$, but we can only say $\|\bar{\pi}_L^{\Gamma_1} g\|_{\overline{M_{\Gamma_1}}} \leq \|\bar{\pi}_L g\|_{\mathcal{V}_{L,\Gamma_1}}$.

Example 6.6. Continuing Example 6.2 yields $H_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega) = H^1_{\Gamma_0}(\Omega)^{m_1} = \{f \in H^1(\Omega)^{m_1} : \mathbb{1}_{\Gamma_1}\gamma_0 f = 0\}$ which already appeared in the proof of Lemma 6.3. Moreover, we have $\bar{\pi}_L = \gamma_0$, $\bar{\pi}_L^{\Gamma_1} = \mathbb{1}_{\Gamma_1}\gamma_0$, $\mathcal{V}_L = H^{1/2}(\partial\Omega)$, and $\mathcal{V}_{L,\Gamma_1} = \{f \in H^{1/2}(\partial\Omega) : f\big|_{\Gamma_0} = 0\}$.

Lemma 6.7. *The space $H_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)$ equipped with $\langle .,.\rangle_{H(L_\partial^{\mathsf{H}},\Omega)}$ is a Hilbert space and $H^1(\Omega)^{m_1} \cap H_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)$ is dense in $H_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)$. Moreover, $\mathcal{V}_{L,\Gamma_1}$ is a closed subspace of $\mathcal{V}_L$ and therefore also a Hilbert space.*

Proof. By definition of $H_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)$ and Lemma 6.4 we have

$$H_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega) = \ker \bar{\pi}_L^{\Gamma_0} = \overline{\ker \pi_L^{\Gamma_0}} = \overline{H^1(\Omega)^{m_1} \cap H_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)}.$$

Note that $\ker \pi_L \subseteq \ker \pi_L^{\Gamma_0}$, since $\pi_L^{\Gamma_0} = \mathbb{1}_{\Gamma_0}\pi_L$. Again by Lemma 6.4, we have

$$\ker \bar{\pi}_L = \overline{\ker \pi_L} \subseteq \overline{\ker \pi_L^{\Gamma_0}} = \ker \bar{\pi}_L^{\Gamma_0}.$$

Therefore, $\bar{\pi}_L^{\Gamma_0} \circ \bar{\pi}_L^{-1} : \mathcal{V}_L \to \overline{M_{\Gamma_0}}$ is single-valued (well-defined). For arbitrary $\phi \in \mathcal{V}_L$ and $g \in \bar{\pi}_L^{-1}\phi$ we have

$$\|\bar{\pi}_L^{\Gamma_0} \circ \bar{\pi}_L^{-1}\phi\|_{\overline{M_{\Gamma_0}}} = \inf_{k \in \ker \bar{\pi}_L^{\Gamma_0}} \|g+k\|_{H(L_\partial^{\mathsf{H}},\Omega)} \leq \inf_{k \in \ker \bar{\pi}_L} \|g+k\|_{H(L_\partial^{\mathsf{H}},\Omega)} = \|\phi\|_{\mathcal{V}_L}.$$

Hence, $\bar{\pi}_L^{\Gamma_0} \circ \bar{\pi}_L^{-1}$ is continuous and $\ker \bar{\pi}_L^{\Gamma_0} \circ \bar{\pi}_L^{-1}$ is closed in $\mathcal{V}_L$ and therefore also a Hilbert space endowed with $\langle .,.\rangle_{\mathcal{V}_L}$. The equivalences

$$\phi \in \ker \bar{\pi}_L^{\Gamma_0} \circ \bar{\pi}_L^{-1} \quad \Leftrightarrow \quad \bar{\pi}_L^{-1}\phi \subseteq \ker \bar{\pi}_L^{\Gamma_0} \quad \Leftrightarrow \quad \phi \in \underbrace{\operatorname{ran} \bar{\pi}_L\big|_{\ker \bar{\pi}_L^{\Gamma_0}}}_{=\mathcal{V}_{L,\Gamma_1}}$$

imply that $\mathcal{V}_{L,\Gamma_1}$ is closed and therefore a Hilbert space. ❑

Proposition 6.8. *The mapping $\mathbb{1}_{\Gamma_1} L_\nu \gamma_0 \colon H^1(\Omega)^{m_2} \to L^2_\pi(\Gamma_1)$ can be extended to a linear continuous mapping*

$$\bar{L}_\nu^{\Gamma_1} \colon H(L_\partial,\Omega) \to \mathcal{V}'_{L,\Gamma_1},$$

such that $\|\bar{L}_\nu^{\Gamma_1} f\|_{\mathcal{V}'_{L,\Gamma_1}} \leq \|f\|_{H(L_\partial,\Omega)}$.

Proof. Let $f \in H^1(\Omega)^{m_2}$. For $g \in H^1(\Omega)^{m_1} \cap H_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega)$ we have by Corollary 3.9

$$\left|\langle \mathbb{1}_{\Gamma_1} L_\nu \gamma_0 f, \bar{\pi}_L g\rangle_{L^2(\Gamma_1)^{m_1}}\right| = \left|\langle L_\nu \gamma_0 f, \bar{\pi}_L g\rangle_{L^2(\partial\Omega)^{m_1}}\right| \leq \|f\|_{H(L_\partial,\Omega)} \|g\|_{H(L_\partial^{\mathsf{H}},\Omega)}.$$

By Lemma 6.7 the subspace $M \coloneqq \operatorname{ran} \bar{\pi}_L\big|_{H^1(\Omega)^{m_1} \cap H_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)} \subseteq L^2_\pi(\Gamma_1)^{m_1}$ of $\mathcal{V}_{L,\Gamma_1}$ is dense in $\mathcal{V}_{L,\Gamma_1}$. For $\phi \in M$ there exists at least one $g \in H^1(\Omega)^{m_1} \cap H_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega)$ such that $\pi_L g = \phi$. Hence, we can rewrite the inequality as

$$\begin{aligned}\left|\langle \mathbb{1}_{\Gamma_1} L_\nu \gamma_0 f, \phi\rangle_{L^2(\Gamma_1)^{m_1}}\right| &\leq \|f\|_{H(L_\partial,\Omega)} \inf_{\substack{g \in H^1(\Omega)^{m_1} \cap H_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega) \\ \bar{\pi}_L g = \phi}} \|g\|_{H(L_\partial^{\mathsf{H}},\Omega)} \\ &= \|f\|_{H(L_\partial,\Omega)} \|\phi\|_{\mathcal{V}_{L,\Gamma_1}}.\end{aligned}$$

We will extend the mapping $\phi \mapsto \langle \mathbb{1}_{\Gamma_1} L_\nu \gamma_0 f, \phi\rangle_{L^2(\Gamma_1)^{m_1}}$ by continuity on $\mathcal{V}_{L,\Gamma_1}$. We will denote this extension by Ξ_f. Therefore, we have

$$|\Xi_f(\phi)| \leq \|f\|_{H(L_\partial,\Omega)} \|\phi\|_{\mathcal{V}_{L,\Gamma_1}}.$$

This means that the mapping $f \mapsto \Xi_f$ from $H^1(\Omega)^{m_2}$ to $\mathcal{V}'_{L,\Gamma_1}$ is continuous, if we endow $H^1(\Omega)^{m_2}$ with $\|.\|_{H(L_\partial,\Omega)}$. Once again, we will extend this mapping by continuity on $H(L_\partial, \Omega)$ and denote it by $\bar{L}_\nu^{\Gamma_1}$. ❑

Instead of writing $\bar{L}_\nu^{\partial\Omega}$ we will just write $\bar{L}_\nu$.

Remark 6.9. Since $\mathcal{V}_{L,\Gamma_1}$ is a subspace of $\mathcal{V}_{L,\partial\Omega} = \mathcal{V}_L$ every element of $\mathcal{V}'_L$ can also be treated as an element of $\mathcal{V}'_{L,\Gamma_1}$. By definition of $\bar{L}_\nu^{\Gamma_1}$ and $\bar{L}_\nu$ it is easy to see that $\bar{L}_\nu^{\Gamma_1} f = \bar{L}_\nu f\big|_{\mathcal{V}_{L,\Gamma_1}}$ or equivalently $\bar{L}_\nu^{\Gamma_1} f$ and $\bar{L}_\nu f$ coincide as elements of $\mathcal{V}'_{L,\Gamma_1}$ for $f \in H(L_\partial, \Omega)$. Hence, we can say $\mathcal{V}'_L\big|_{\mathcal{V}_{L,\Gamma_1}} \subseteq \mathcal{V}'_{L,\Gamma_1}$. Since Hahn-Banach gives the reverse inclusion we can even say $\mathcal{V}'_L\big|_{\mathcal{V}_{L,\Gamma_1}} = \mathcal{V}'_{L,\Gamma_1}$.

The reason for even defining $\bar{L}_\nu^{\Gamma_1}$ instead of just using $\bar{L}_\nu$ is that the range of its restriction to $H^1(\Omega)^{m_2}$ is also contained in $L^2_\pi(\Gamma_1)$, which will be important for getting a quasi Gelfand triple.

Corollary 6.10. *For $f \in H(L_\partial, \Omega)$ and $g \in H_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega)$ we have*

$$\langle L_\partial f, g\rangle_{L^2(\Omega)^{m_1}} + \langle f, L_\partial^{\mathsf{H}} g\rangle_{L^2(\Omega)^{m_2}} = \langle \bar{L}_\nu f, \bar{\pi}_L g\rangle_{\mathcal{V}'_{L,\Gamma_1}, \mathcal{V}_{L,\Gamma_1}}.$$

For $f \in H(L_\partial, \Omega)$ and $g \in H(L_\partial^{\mathsf{H}}, \Omega)$ we have

$$\begin{aligned}\langle L_\partial f, g\rangle_{L^2(\Omega)^{m_1}} + \langle f, L_\partial^{\mathsf{H}} g\rangle_{L^2(\Omega)^{m_2}} &= \langle \bar{L}_\nu f, \bar{\pi}_L g\rangle_{\mathcal{V}'_L, \mathcal{V}_L} \\ &= \langle \bar{\pi}_{L^{\mathsf{H}}} f, \bar{L}_\nu^{\mathsf{H}} g\rangle_{\mathcal{V}_{L^{\mathsf{H}}}, \mathcal{V}'_{L^{\mathsf{H}}}}.\end{aligned}$$

Proof. Since $H^1(\Omega)^{m_2}$ is dense in $H(L_\partial, \Omega)$ and $H^1(\Omega)^{m_1} \cap H_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega)$ is dense in $H_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega)$, the first equation follows from (6.1) by continuity. The second equation is just the special case $\Gamma_0 = \emptyset$ and switching the roles of L_∂ and L_∂^{H} yields the last equation. ❑

Theorem 6.11. *The mapping $\bar{L}_\nu \colon H(L_\partial, \Omega) \to \mathcal{V}'_L$ is linear, bounded and onto.*

Proof. By Proposition 6.8 we already know that $\bar{L}_\nu$ is linear and bounded from $H(L_\partial, \Omega)$ to $\mathcal{V}'_L$.

Let $\mu \in \mathcal{V}'_L$ be arbitrary. Since $\bar{\pi}_L$ is continuous from $H(L_\partial^{\mathsf{H}}, \Omega)$ to $\mathcal{V}_L$, the mapping $g \mapsto \langle \mu, \bar{\pi}_L g\rangle_{\mathcal{V}'_L, \mathcal{V}_L}$ is continuous from $H(L_\partial^{\mathsf{H}}, \Omega)$ to $\mathbb{C}$. Consequently, there exists an $h \in H(L_\partial^{\mathsf{H}}, \Omega)$ such that

$$\langle h, g\rangle_{H(L_\partial^{\mathsf{H}},\Omega)} = \langle \mu, \pi_L g\rangle_{\mathcal{V}'_L, \mathcal{V}_L} \quad \text{for all} \quad g \in H(L_\partial^{\mathsf{H}}, \Omega).$$

For a test function $v \in \mathcal{D}(\Omega)^{m_1}$ we have

$$\begin{aligned}0 &= \langle \mu, \pi_L v\rangle_{\mathcal{V}_L', \mathcal{V}_L} = \langle h, v\rangle_{H(L_\partial^{\mathsf{H}}, \Omega)} = \langle h, v\rangle_{L^2(\Omega)^{m_1}} + \langle L_\partial^{\mathsf{H}} h, L_\partial^{\mathsf{H}} v\rangle_{L^2(\Omega)^{m_2}} \\ &= \langle h, v\rangle_{\mathcal{D}'(\Omega)^{m_1}, \mathcal{D}(\Omega)^{m_1}} + \big\langle L_\partial^{\mathsf{H}} h, L_\partial^{\mathsf{H}} v\big\rangle_{\mathcal{D}'(\Omega)^{m_2}, \mathcal{D}(\Omega)^{m_2}} \\ &= \big\langle (\mathrm{I} - L_\partial L_\partial^{\mathsf{H}}) h, v\big\rangle_{\mathcal{D}'(\Omega)^{m_1}, \mathcal{D}(\Omega)^{m_1}}.\end{aligned}$$

This means $L_\partial L_\partial^{\mathsf{H}} h = h$ in the sense of distributions. However, $h \in H(L_\partial^{\mathsf{H}}, \Omega)$ implies $h \in L^2(\Omega)$, which in turn gives $L_\partial L_\partial^{\mathsf{H}} h \in L^2(\Omega)^{m_1}$, and $L_\partial^{\mathsf{H}} h \in L^2(\Omega)^{m_2}$. Therefore, $f := L_\partial h \in H(L_\partial, \Omega)$. By Corollary 6.10 for $f = L_\partial^{\mathsf{H}} h \in H(L_\partial, \Omega)$ and $g \in H(L_\partial^{\mathsf{H}}, \Omega)$ we have

$$\begin{aligned}\langle \mu, \pi_L g\rangle_{\mathcal{V}_L', \mathcal{V}_L} &= \langle h, g\rangle_{H(L_\partial^{\mathsf{H}}, \Omega)} = \langle h, g\rangle_{L^2(\Omega)^{m_1}} + \langle L_\partial^{\mathsf{H}} h, L_\partial^{\mathsf{H}} g\rangle_{L^2(\Omega)^{m_2}} \\ &= \langle (\mathrm{I} - L_\partial L_\partial^{\mathsf{H}}) h, g\rangle_{L^2(\Omega)^{m_1}} + \langle \bar{L}_\nu L_\partial^{\mathsf{H}} h, \pi_L g\rangle_{\mathcal{V}_L', \mathcal{V}_L} \\ &= \big\langle \bar{L}_\nu \underbrace{(L_\partial^{\mathsf{H}} h)}_{=f}, \pi_L g\big\rangle_{\mathcal{V}_L', \mathcal{V}_L}.\end{aligned}$$

Hence, $\bar{L}_\nu f = \mu$ and $\bar{L}_\nu$ is onto. ❑

Corollary 6.12. *The mapping $\bar{L}_\nu^{\Gamma_1} : H(L_\partial, \Omega) \to \mathcal{V}_{L,\Gamma_1}'$ is linear, bounded and onto.*

Proof. By Proposition 6.8 we already know that $\bar{L}_\nu^{\Gamma_1}$ is linear and bounded form $H(L_\partial, \Omega)$ to $\mathcal{V}_L'$. Remark 6.9 gives $\bar{L}_\nu f\big|_{\mathcal{V}_{L,\Gamma_1}} = \bar{L}_\nu^{\Gamma_1} f$ for $f \in H(L_\partial, \Omega)$ and $\mathcal{V}_{L,\Gamma_1}' = \mathcal{V}_L'\big|_{\mathcal{V}_{L,\Gamma_1}}$, which completes the proof. ❑

Theorem 6.13. *$(\mathcal{V}_{L,\Gamma_1}, L^2_\pi(\Gamma_1), \mathcal{V}_{L,\Gamma_1}')$ is a quasi Gelfand triple.*

Proof. Let $\tilde{D}_+ := \operatorname{ran} \pi_L\big|_{H^1_{\Gamma_0}(\Omega)^{m_1}}$ equipped with $\|.\|_{\mathcal{X}_+} = \|.\|_{\mathcal{V}_{L,\Gamma_1}}$ and let D_- denote the corresponding set from Definition 5.2 with $\mathcal{X}_0 = L^2_\pi(\Gamma_1)$. Then by Remark 5.3 $\|g\|_{\mathcal{X}_-} = \|g\|_{\mathcal{V}_{L,\Gamma_1}'}$ for $g \in D_-$ and $\operatorname{ran} \mathbb{1}_{\Gamma_1} L_\nu \gamma_0 \subseteq D_-$ (by Proposition 6.8). By definition $\operatorname{ran} \mathbb{1}_{\Gamma_1} L_\nu \gamma_0$ is dense in $L^2_\pi(\Gamma_1)$ and by Proposition 6.8 and Corollary 6.12 also dense in $\mathcal{V}_{L,\Gamma_1}'$. Consequently, also D_- is dense in both $L^2_\pi(\Gamma_1)$ and $\mathcal{V}_{L,\Gamma_1}'$. Hence, assertion (iv) of Proposition 5.8 is satisfied, and by Remark 5.18 the completions of $\tilde{D}_+$ and D_- form a quasi Gelfand triple with pivot space $L^2_\pi(\Gamma_1)$. By construction the completion of $\tilde{D}_+$ is $\mathcal{V}_{L,\Gamma_1}$. By the density of D_- in $\mathcal{V}_{L,\Gamma_1}'$ and $\|g\|_{\mathcal{X}_-} = \|g\|_{\mathcal{V}_{L,\Gamma_1}'}$ for $g \in D_-$ the completion of D_- is $\mathcal{V}_{L,\Gamma_1}'$. ❑

Corollary 6.14. *$H_0(L_\partial^{\mathsf{H}}, \Omega) = H_{\partial\Omega}(L_\partial^{\mathsf{H}}, \Omega) = \ker \bar{\pi}_L = \ker \bar{L}_\nu^{\mathsf{H}}$ and $H_0(L_\partial, \Omega) = H_{\partial\Omega}(L_\partial, \Omega) = \ker \bar{\pi}_{L^{\mathsf{H}}} = \ker \bar{L}_\nu$.*

Proof. For $g \in H_0(L_\partial^{\mathsf{H}}, \Omega)$ there is a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\Omega)$ converging to g, which implies $\bar{\pi}_L g = \lim_{n\to\infty} \bar{\pi}_L g_n = 0$. Therefore, $H_0(L_\partial^{\mathsf{H}}, \Omega) \subseteq \ker \bar{\pi}_L = H_{\partial\Omega}(L_\partial^{\mathsf{H}}, \Omega)$. On the other hand, if $g \in H_{\partial\Omega}(L_\partial^{\mathsf{H}}, \Omega)$, then

$$\langle L_\partial f, g\rangle_{L^2(\Omega)^{m_1}} + \langle f, L_\partial^{\mathsf{H}} g\rangle_{L^2(\Omega)^{m_2}} = \langle \bar{L}_\nu f, \bar{\pi}_L g\rangle_{\mathcal{V}_L', \mathcal{V}_L} = 0$$

for all $f \in H(L_\partial,\Omega)$. Hence, by Lemma 3.17 $g \in H_0(L_\partial^{\mathsf{H}},\Omega)$. Consequently, $H_0(L_\partial^{\mathsf{H}},\Omega) = H_{\partial\Omega}(L_\partial^{\mathsf{H}},\Omega)$. The second equality of the statement holds by definition and the third will be proven by the following equivalences

$$\begin{aligned} g \in \ker \pi_L &\Leftrightarrow \langle \bar\pi_L g, \psi\rangle_{\mathcal{V}_L',\mathcal{V}_L} = 0 \quad \text{for all} \quad \psi \in \mathcal{V}_L' \\ &\Leftrightarrow \langle \bar\pi_L g, \bar L_\nu f\rangle_{\mathcal{V}_L',\mathcal{V}_L} = 0 \quad \text{for all} \quad f \in H(L_\partial,\Omega) \\ &\overset{C.6.10}{\Leftrightarrow} \langle \bar L_\nu^{\mathsf{H}} g, \bar\pi_{L^{\mathsf{H}}} f\rangle_{\mathcal{V}_{L^{\mathsf{H}}}',\mathcal{V}_{L^{\mathsf{H}}}} = 0 \quad \text{for all} \quad f \in H(L_\partial,\Omega) \\ &\Leftrightarrow \langle \bar L_\nu^{\mathsf{H}} g, \phi\rangle_{\mathcal{V}_{L^{\mathsf{H}}}',\mathcal{V}_{L^{\mathsf{H}}}} = 0 \quad \text{for all} \quad \phi \in \mathcal{V}_{L^{\mathsf{H}}} \\ &\Leftrightarrow g \in \ker \bar L_\nu^{\mathsf{H}}. \end{aligned}$$

Switching L with L^{H} yields $H_0(L_\partial,\Omega) = H_{\partial\Omega}(L_\partial,\Omega) = \ker \bar\pi_{L^{\mathsf{H}}} = \ker \bar L_\nu$. ❑

7. **Existence and uniqueness via boundary triples.** In this section we will show that there is a boundary triple associated to the port-Hamiltonian differential operator $(P_\partial + P_0)\mathcal{H}$, which enables us to formulate boundary conditions that admit existence and uniqueness of solutions. Moreover, we will parameterize all boundary conditions that provide unique solutions that are non-increasing in the Hamiltonian.

Recall the setting in section 4. Using $\pi_P = \begin{bmatrix} \pi_L & 0 \\ 0 & \pi_{L^{\mathsf{H}}} \end{bmatrix}$, Lemma 6.4 and Proposition 6.8, it is easy to see that $\mathcal{V}_P = \mathcal{V}_L \times \mathcal{V}_{L^{\mathsf{H}}}$ and therefore $\mathcal{V}_P' = \mathcal{V}_L' \times \mathcal{V}_{L^{\mathsf{H}}}'$. Furthermore, for $\bar P_\nu \colon H(P_\partial,\Omega) \to \mathcal{V}_P'$ and $\bar\pi_P \colon H(P_\partial,\Omega) \to \mathcal{V}_P$ we have

$$\bar P_\nu = \begin{bmatrix} 0 & \bar L_\nu \\ \bar L_\nu^{\mathsf{H}} & 0 \end{bmatrix} \quad \text{and} \quad \bar\pi_P = \begin{bmatrix} \bar\pi_L & 0 \\ 0 & \bar\pi_{L^{\mathsf{H}}} \end{bmatrix}.$$

Recall the splitting $x = \left[\begin{smallmatrix} x_{L^{\mathsf{H}}} \\ x_L \end{smallmatrix}\right]$. Accordingly, we introduce $\mathcal{H}x = \begin{bmatrix} (\mathcal{H}x)_{L^{\mathsf{H}}} \\ (\mathcal{H}x)_L \end{bmatrix}$ for $x \in \mathcal{H}^{-1}(H(P_\partial,\Omega))$, so that

$$P_\partial \mathcal{H}x = \begin{bmatrix} L_\partial(\mathcal{H}x)_L \\ L_\partial^{\mathsf{H}}(\mathcal{H}x)_{L^{\mathsf{H}}} \end{bmatrix}, \quad \begin{bmatrix} 0 & \bar L_\nu \end{bmatrix} \mathcal{H}x = \bar L_\nu (\mathcal{H}x)_L, \quad \begin{bmatrix} \bar\pi_L & 0 \end{bmatrix} \mathcal{H}x = \bar\pi_L (\mathcal{H}x)_{L^{\mathsf{H}}}.$$

Theorem 7.1. *The operator*

$$A_0 \coloneqq -(P_\partial + P_0)\mathcal{H}, \quad \operatorname{dom} A_0 \coloneqq \mathcal{H}^{-1}(\ker \bar P_\nu)$$

is closed, skew-symmetric, and densely defined on $\mathcal{X}_\mathcal{H}$. Its adjoint is

$$A_0^* = (P_\partial + P_0)\mathcal{H}, \quad \operatorname{dom} A_0^* = \mathcal{H}^{-1}(H(P_\partial,\Omega)).$$

Let $B_1 = \begin{bmatrix} \bar\pi_L & 0 \end{bmatrix} \mathcal{H}$, $B_2 = \begin{bmatrix} 0 & \bar L_\nu \end{bmatrix} \mathcal{H}$ and Ψ be the duality map of $(\mathcal{V}_L, L_\pi^2(\partial\Omega), \mathcal{V}_L')$. Then $(\mathcal{V}_L, B_1, \Psi B_2)$ is a boundary triple for A_0^*.

Proof. We define $\tilde A$ as $(P_\partial + P_0)\mathcal{H}$ with $\operatorname{dom} \tilde A = \mathcal{H}^{-1}(H(P_\partial,\Omega))$ on $\mathcal{X}_\mathcal{H}$. By Lemma 3.5 $P_\partial \colon H(P_\partial,\Omega) \subseteq L^2(\Omega)^m \to L^2(\Omega)^m$ is a closed operator. Since $\mathcal{H}$ is a bounded operator on $L^2(\Omega)^m$, and $\mathcal{X}_\mathcal{H}$ and $L^2(\Omega)^m$ have equivalent norms, it is easy to see that $\tilde A \colon \mathcal{H}^{-1}(H(P_\partial,\Omega)) \subseteq \mathcal{X}_\mathcal{H} \to \mathcal{X}_\mathcal{H}$ is closed. Let B^{*_H} denote the adjoint of B with respect to $\langle .,. \rangle_H$ for any Hilbert space H. The adjoint of $\tilde A$ can be calculated by

$$\tilde A^* = \big((P_\partial + P_0)\mathcal{H}\big)^{*_{\mathcal{X}_\mathcal{H}}} = \mathcal{H}^{-1}\big((P_\partial + P_0)\mathcal{H}\big)^{*_{L^2}} \mathcal{H} = (P_\partial^{*_{L^2}} + P_0^{*_{L^2}})\mathcal{H}$$

and according to Remark 3.7 we have $P_\partial^{*_{L^2}} = -P_\partial|_{\operatorname{dom} P_\partial^{*_{L^2}}}$ where $\operatorname{dom} P_\partial^{*_{L^2}} \subseteq H(P_\partial,\Omega)$. Hence,

$$\tilde A^* = -(P_\partial + P_0)\mathcal{H}|_{\mathcal{H}^{-1}(\operatorname{dom} P_\partial^{*_{L^2}})} = -\tilde A|_{\mathcal{H}^{-1}(\operatorname{dom} P_\partial^{*_{L^2}})} \subseteq -\tilde A.$$

Consequently, $\tilde{A}^*$ is skew-symmetric on $\mathcal{X}_{\mathcal{H}}$. Since $\tilde{A}$ is closed, we have $\tilde{A}^{**} = \tilde{A}$.

Now we know that $\tilde{A}$ is the adjoint of a skew-symmetric operator. So we can talk about boundary triples for $\tilde{A}$. First we note that

$$\operatorname{ran} \left[\begin{smallmatrix} B_1 \\ \Psi B_2 \end{smallmatrix}\right] = \operatorname{ran} \bar{\pi}_L \times \operatorname{ran} \Psi \bar{L}_\nu = \mathcal{V}_L \times \mathcal{V}_L.$$

Since $\mathcal{H}$ is self-adjoint and P_0 is skew-adjoint, we have for $x, y \in \operatorname{dom} \tilde{A}$

$$\begin{aligned} &\langle \tilde{A}x, y\rangle_{\mathcal{X}_{\mathcal{H}}} + \langle x, \tilde{A}y\rangle_{\mathcal{X}_{\mathcal{H}}} \\ &\qquad = \langle P_\partial \mathcal{H}x, \mathcal{H}y\rangle_{L^2} + \langle \mathcal{H}x, P_\partial \mathcal{H}y\rangle_{L^2} \end{aligned}$$

by the the identity $P_\partial = \left[\begin{smallmatrix} 0 & L_\partial \\ L_\partial^{\mathsf{H}} & 0 \end{smallmatrix}\right]$ and Corollary 6.10 we further have

$$\begin{aligned} &= \left\langle \begin{bmatrix} L_\partial (\mathcal{H}x)_L \\ L_\partial^{\mathsf{H}} (\mathcal{H}x)_{L^{\mathsf{H}}} \end{bmatrix}, \begin{bmatrix} (\mathcal{H}y)_{L^{\mathsf{H}}} \\ (\mathcal{H}y)_L \end{bmatrix} \right\rangle_{L^2} + \left\langle \begin{bmatrix} (\mathcal{H}x)_{L^{\mathsf{H}}} \\ (\mathcal{H}x)_L \end{bmatrix}, \begin{bmatrix} L_\partial (\mathcal{H}y)_L \\ L_\partial^{\mathsf{H}} (\mathcal{H}y)_{L^{\mathsf{H}}} \end{bmatrix} \right\rangle_{L^2} \\ &= \langle L_\partial (\mathcal{H}x)_L, (\mathcal{H}y)_{L^{\mathsf{H}}}\rangle_{L^2} + \left\langle (\mathcal{H}x)_L, L_\partial^{\mathsf{H}} (\mathcal{H}y)_{L^{\mathsf{H}}}\right\rangle_{L^2} \\ &\qquad + \left\langle L_\partial^{\mathsf{H}} (\mathcal{H}x)_{L^{\mathsf{H}}}, (\mathcal{H}y)_L\right\rangle_{L^2} + \left\langle (\mathcal{H}x)_{L^{\mathsf{H}}}, L_\partial^{\mathsf{H}} (\mathcal{H}y)_L\right\rangle_{L^2} \\ &= \langle \bar{L}_\nu (\mathcal{H}x)_L, \bar{\pi}_L (\mathcal{H}y)_{L^{\mathsf{H}}}\rangle_{\mathcal{V}_L', \mathcal{V}_L} + \langle \bar{\pi}_L (\mathcal{H}x)_{L^{\mathsf{H}}}, \bar{L}_\nu (\mathcal{H}y)_L\rangle_{\mathcal{V}_L, \mathcal{V}_L'} \\ &= \langle \Psi B_2 x, B_1 y\rangle_{\mathcal{V}_L} + \langle B_1 x, \Psi B_2 y\rangle_{\mathcal{V}_L}. \end{aligned}$$

Therefore, $(\mathcal{V}_L, B_1, \Psi B_2)$ is a boundary triple for $\tilde{A}$.

By Lemma 2.2 $\operatorname{dom} \tilde{A}^* = \ker B_1 \cap \ker B_2$, which is equal to

$$\ker B_1 \cap \ker B_2 = \mathcal{H}^{-1}\big(\ker \begin{bmatrix} \bar{\pi}_L & 0 \end{bmatrix} \cap \ker \begin{bmatrix} 0 & \bar{L}_\nu \end{bmatrix}\big) = \mathcal{H}^{-1}\big(\ker \bar{\pi}_L \times \ker \bar{L}_\nu\big).$$

By Corollary 6.14 this is equal to $\mathcal{H}^{-1}(\ker \bar{L}_\nu^{\mathsf{H}} \times \ker \bar{L}_\nu) = \mathcal{H}^{-1}(\ker \bar{P}_\nu)$. Hence, $\tilde{A}^* = A_0$ and $A_0^* = \tilde{A}$. ❑

Remark 7.2. We can replace $(\mathcal{V}_L, B_1, \Psi B_2)$ by $(\mathcal{V}_L', \Psi^* B_1, B_2)$ in the previous theorem.

Theorem 7.3. *Let A_0^* be the operator from the previous theorem and Ψ_{Γ_1} the duality map associated to the quasi Gelfand triple $(\mathcal{V}_{L,\Gamma_1}, L^2_\pi(\Gamma_1), \mathcal{V}_{L,\Gamma_1}')$. Then we have $(\mathcal{V}_{L,\Gamma_1}, \begin{bmatrix} \bar{\pi}_L & 0 \end{bmatrix} \mathcal{H}, \Psi_{\Gamma_1} \begin{bmatrix} 0 & \bar{L}_\nu^{\Gamma_1} \end{bmatrix} \mathcal{H})$ as a boundary triple for*

$$A := A_0^*\big|_{\mathcal{H}^{-1}\left(H_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega) \times H(L_\partial, \Omega)\right)}.$$

Proof. Since we already have a boundary triple for A_0^*, we can show that A is the adjoint of a skew-symmetric operator by Proposition 2.3 (iii). Hence, we have to check, whether $\left[\begin{smallmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{smallmatrix}\right] \mathcal{C}^\perp \subseteq \mathcal{C}$ in $\mathcal{V}_L \times \mathcal{V}_L$, where $\mathcal{C}$ is the corresponding relation to the domain of A according to Proposition 2.3. For B_1, B_2 being the mappings from the previous theorem we have (Note that $\mathcal{V}_{L,\Gamma_1}$ is a subspace of $\mathcal{V}_L$; Lemma 6.7)

$$\mathcal{C} = \begin{bmatrix} B_1 \\ \Psi B_2 \end{bmatrix} \operatorname{dom} A = \mathcal{V}_{L,\Gamma_1} \times \mathcal{V}_L$$

$$\begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix} \mathcal{C}^\perp = \{0\} \times \mathcal{V}_{L,\Gamma_1}^\perp \subseteq \mathcal{V}_{L,\Gamma_1} \times \mathcal{V}_L = \mathcal{C}.$$

For $x, y \in \operatorname{dom} A$ we have, using Remark 6.9,

$$\begin{aligned} \langle B_1 x, \Psi B_2 y\rangle_{\mathcal{V}_L} &= \left\langle \bar{\pi}_L (\mathcal{H}x)_{L^{\mathsf{H}}}, \bar{L}_\nu (\mathcal{H}y)_L\right\rangle_{\mathcal{V}_L, \mathcal{V}_L'} = \left\langle \bar{\pi}_L (\mathcal{H}x)_{L^{\mathsf{H}}}, \bar{L}_\nu^{\Gamma_1} (\mathcal{H}y)_L\right\rangle_{\mathcal{V}_{L,\Gamma_1}, \mathcal{V}_{L,\Gamma_1}'} \\ &= \left\langle \begin{bmatrix} \bar{\pi}_L & 0 \end{bmatrix} \mathcal{H}x, \Psi_{\Gamma_1} \begin{bmatrix} 0 & \bar{L}_\nu^{\Gamma_1} \end{bmatrix} \mathcal{H}y\right\rangle_{\mathcal{V}_{L,\Gamma_1}}, \end{aligned}$$

which yields item (ii) in Definition 2.1. By $\operatorname{ran} \left[\begin{smallmatrix} \bar{\pi}_L & 0 \\ 0 & \Psi_{\Gamma_1} \bar{L}_\nu^{\Gamma_1} \end{smallmatrix} \right] \Big|_{H_{\Gamma_0}(L_\partial^{\mathsf{H}},\Omega)\times H(L_\partial,\Omega)} = \mathcal{V}_{L,\Gamma_1} \times \mathcal{V}_{L,\Gamma_1}$, the remaining item (i) is fulfilled. ❑

The next theorem is [9, Theorem 2.5].

Theorem 7.4. *Let A_0 be a skew-symmetric operator on a Hilbert space X and $(\mathcal{B}, B_1, B_2)$ be a boundary triple for A_0^*. Furthermore let $\mathcal{K}$ be a Hilbert space, $W_B = \begin{bmatrix} W_1 & W_2 \end{bmatrix}$, where $W_1, W_2 \in \mathcal{L}(\mathcal{B}, \mathcal{K})$, and $A \coloneqq A_0^*\big|_{\operatorname{dom} A}$, where $\operatorname{dom} A = \ker W_B \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$. If $\operatorname{ran} W_1 - W_2 \subseteq \operatorname{ran} W_1 + W_2$ then the following assertions are equivalent.*

(i) *The operator A generates a contraction semigroup on X.*
(ii) *The operator A is dissipative.*
(iii) *The operator $W_1 + W_2$ is injective and the following operator inequality holds*
$$W_1 W_2^* + W_2 W_1^* \geq 0.$$

We will reformulate this theorem to fit our situation.

Corollary 7.5. *Let $\mathcal{K}$ be some Hilbert space and $W = \begin{bmatrix} W_1 & W_2 \end{bmatrix} : \mathcal{V}_{L,\Gamma_1} \times \mathcal{V}_{L,\Gamma_1} \to \mathcal{K}$ a bounded linear mapping such that $\operatorname{ran} W_1 - W_2 \subseteq \operatorname{ran} W_1 + W_2$. Let*
$$D \coloneqq \Big\{ x \in \mathcal{H}^{-1}(H_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega) \times H(L_\partial, \Omega)) \\ : W_1 \begin{bmatrix} \bar{\pi}_L & 0 \end{bmatrix} \mathcal{H} x + W_2 \Psi \begin{bmatrix} 0 & \bar{L}_\nu^{\Gamma_1} \end{bmatrix} \mathcal{H} x = 0 \Big\},$$
where $\Psi : \mathcal{V}_{L,\Gamma_1}' \to \mathcal{V}_{L,\Gamma_1}$ is the duality mapping corresponding to the quasi Gelfand triple. Then the following assertions are equivalent.

(i) *$(P_\partial + P_0)\mathcal{H}\big|_D$ generates a contraction semigroup.*
(ii) *$(P_\partial + P_0)\mathcal{H}\big|_D$ is dissipative.*
(iii) *The operator $W_1 + W_2$ is injective and the following operator inequality holds*
$$W_1 W_2^* + W_2 W_1^* \geq 0.$$

Corollary 7.5 already gives a parameterization via W for all boundary conditions that make $(P_\partial + P_0)\mathcal{H}$ a generator of a contraction semigroup. In particular the corresponding PDEs have unique solutions that continuously depend on the initial state and don't grow in the Hamiltonian. However, checking continuity for boundary operators which map into $\mathcal{V}_L$ can be difficult. Hence, it would be appreciated to reduce the conditions on the boundary operators to conditions on better known spaces like the pivot space $L^2(\partial\Omega)$. The next theorem will provide this.

The following result is a generalization of [9, Theorem 2.6] for quasi Gelfand triple and also fixes some minor issues, like the specific choice of Ψ and the closedness of $\begin{bmatrix} V_1\big|_{\mathcal{B}_+ \cap \mathcal{B}_0} & V_2\big|_{\mathcal{B}_- \cap \mathcal{B}_0} \end{bmatrix}$ as an operator from $\mathcal{B}_+ \times \mathcal{B}_-$ to $\mathcal{K}$.

Theorem 7.6. *Let $(\mathcal{B}_+, \mathcal{B}_0, \mathcal{B}_-)$ be a quasi Gelfand triple, A_0 be a closed skew-symmetric operator and $(\mathcal{B}_+, B_1, \Psi B_2)$ be a boundary triple for A_0^*, where Ψ is the duality map of the quasi Gelfand triple. For $V_1, V_2 \in \mathcal{L}(\mathcal{B}_0, \mathcal{K})$ we define*
$$D \coloneqq \left\{ a \in \operatorname{dom} A_0^* : B_1 a, B_2 a \in \mathcal{B}_0 \ \text{ and } \ \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} a = 0 \right\}$$
and the operator $A \coloneqq A_0^*\big|_D$. If

(i) *$\begin{bmatrix} V_1\big|_{\mathcal{B}_0 \cap \mathcal{B}_+} & V_2\big|_{\mathcal{B}_0 \cap \mathcal{B}_-} \end{bmatrix}$ is closed as an operator from $\mathcal{B}_+ \times \mathcal{B}_-$ to $\mathcal{K}$,*

(ii) $\ker \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ *is dissipative as linear relation on* $\mathcal{B}_0$,

(iii) $V_1 V_2^* + V_2 V_1^* \geq 0$ *as operator on* $\mathcal{K}$,

then A *is a generator of a contraction semigroup.*

Proof. It is sufficient to show that A is closed, and A and A^* are dissipative.

Step 1. Showing that A *is closed and dissipative.* We have

$$a \in D \Leftrightarrow \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} a \in (\mathcal{B}_0 \times \mathcal{B}_0) \cap \ker \begin{bmatrix} V_1 & V_2 \end{bmatrix}$$
$$\Leftrightarrow \begin{bmatrix} B_1 \\ \Psi B_2 \end{bmatrix} a \in \underbrace{\ker \begin{bmatrix} V_1|_{\mathcal{B}_0 \cap \mathcal{B}_+} & V_2 \Psi^*|_{\Psi(\mathcal{B}_0 \cap \mathcal{B}_-)} \end{bmatrix}}_{=:\mathcal{C}}.$$

We can write

$$\mathcal{C} = \left\{ \begin{bmatrix} q \\ p \end{bmatrix} \in \mathcal{B}_+ \times \mathcal{B}_+ : q \in \mathcal{B}_0,\ \exists \tilde{p} \in \mathcal{B}_0 : p = \Psi \tilde{p},\ V_1 q + V_2 \Psi^* p = 0 \right\}.$$

For $\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right] \in \mathcal{C}$ we have

$$\operatorname{Re}\langle q, p \rangle_{\mathcal{B}_+} = \operatorname{Re}\langle q, \Psi \tilde{p} \rangle_{\mathcal{B}_+} = \operatorname{Re}\langle q, \tilde{p} \rangle_{\mathcal{B}_+, \mathcal{B}_-} = \operatorname{Re}\langle q, \tilde{p} \rangle_{\mathcal{B}_0} \leq 0,$$

which implies the dissipativity of A by Proposition 2.3. Assumption (i) implies that $\mathcal{C}$ is closed in $\mathcal{B}_+^2$, which implies the closedness of A by Proposition 2.3.

Step 2. Showing that A^* *is dissipative.* By Proposition 2.3 we can characterize the domain of A^* by

$$d \in \operatorname{dom} A^* \Leftrightarrow \begin{bmatrix} B_1 \\ \Psi B_2 \end{bmatrix} d \in \begin{bmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix} \mathcal{C}^{\perp_{\mathcal{B}_+^2}}$$
$$\Leftrightarrow \begin{bmatrix} \Psi B_2 \\ B_1 \end{bmatrix} d \in \overline{\operatorname{ran} \begin{bmatrix} \left(V_1|_{\mathcal{B}_0 \cap \mathcal{B}_+}\right)^{*_{\mathcal{B}_+}} \\ \left(V_2 \Psi^*|_{\Psi(\mathcal{B}_0 \cap \mathcal{B}_-)}\right)^{*_{\mathcal{B}_+}} \end{bmatrix}}^{\mathcal{B}_+^2}.$$

The second equivalence needed the closedness in assumption (i), since $(\ker T)^\perp = \overline{\operatorname{ran} T^*}$ for a linear relation (or even unbounded operator) T is not true in general. Note that if P is a bounded and everywhere defined operator, and Q is a linear relation, then $(PQ)^* = Q^* P^*$. Hence, by Proposition 5.16

$$\left(V_1|_{\mathcal{B}_0 \cap \mathcal{B}_+}\right)^{*_{\mathcal{B}_+}} = (V_1 \iota_+)^* = \iota_+^* V_1^* = \Psi V_1^*|_{V_1^{*-1}(\mathcal{B}_0 \cap \mathcal{B}_-)},$$

where $\iota_+ : \mathcal{B}_+ \cap \mathcal{B}_0 \subseteq \mathcal{B}_+ \to \mathcal{B}_0$ is one embedding of the quasi Gelfand triple and

$$\left(V_2 \Psi^*|_{\Psi(\mathcal{B}_0 \cap \mathcal{B}_-)}\right)^{*_{\mathcal{B}_+}} = (V_2 \iota_- \Psi^*)^* = (\iota_- \Psi^*)^* V_2^*,$$

where $\iota_- : \mathcal{B}_- \cap \mathcal{B}_0 \subseteq \mathcal{B}_- \to \mathcal{B}_0$ is the other embedding of the quasi Gelfand triple. From $(\Psi \iota_-^*)^* = \iota_- \Psi^*$ and $\iota_-^* = \Psi^* \iota_+^{-1}$ (Proposition 5.16) follows $(\iota_- \Psi^*)^* = \overline{\Psi \iota_-^*} = \iota_+^{-1}$. Consequently,

$$\left(V_2 \Psi^*|_{\Psi(\mathcal{B}_0 \cap \mathcal{B}_-)}\right)^{*_{\mathcal{B}_+}} = \iota_+^{-1} V_2^* = V_2^*|_{V_2^{*-1}(\mathcal{B}_0 \cap \mathcal{B}_+)}.$$

Hence, for

$$\begin{bmatrix} x \\ y \end{bmatrix} \in \operatorname{ran} \begin{bmatrix} \left(V_1|_{\mathcal{B}_0 \cap \mathcal{B}_+}\right)^{*_{\mathcal{B}_+}} \\ \left(V_2 \Psi^*|_{\Psi(\mathcal{B}_0 \cap \mathcal{B}_-)}\right)^{*_{\mathcal{B}_+}} \end{bmatrix}$$
$$= \left\{ \begin{bmatrix} \Psi V_1^* \\ V_2^* \end{bmatrix} k : k \in V_1^{*-1}(\mathcal{B}_0 \cap \mathcal{B}_-) \cap V_2^{*-1}(\mathcal{B}_0 \cap \mathcal{B}_+) \right\},$$

we have

$$\begin{aligned}\operatorname{Re}\langle x,y\rangle_{\mathcal{B}_+} &= \operatorname{Re}\langle \Psi V_1^* k, V_2^* k\rangle_{\mathcal{B}_+} = \operatorname{Re}\langle V_1^* k, V_2^* k\rangle_{\mathcal{B}_-,\mathcal{B}_+} = \operatorname{Re}\langle V_1^* k, V_2^* k\rangle_{\mathcal{B}_0} \\ &= \operatorname{Re}\langle V_2 V_1^* k, k\rangle_{\mathcal{K}} \geq 0.\end{aligned}$$

Therefore, $\mathcal{C}^\perp$ is accretive and by Proposition 2.3 also $A_0\big|_{\operatorname{dom} A^*}$ is accretive, which yields $A^* = -A_0\big|_{\operatorname{dom} A^*}$ is dissipative. ❑

Remark 7.7. If we are already satisfied with the operator closure $\overline{A}$ is a generator (instead of A) in the previous theorem, then we can replace condition (i) by

$$\ker \overline{\begin{bmatrix} V_1\big|_{\mathcal{B}_0\cap\mathcal{B}_+} & V_2\big|_{\mathcal{B}_0\cap\mathcal{B}_-}\end{bmatrix}} \subseteq \overline{\ker \begin{bmatrix} V_1\big|_{\mathcal{B}_0\cap\mathcal{B}_+} & V_2\big|_{\mathcal{B}_0\cap\mathcal{B}_-}\end{bmatrix}}^{\mathcal{B}_+\times\mathcal{B}_-}, \tag{7.1}$$

where $\overline{\begin{bmatrix} V_1|_{\mathcal{B}_0\cap\mathcal{B}_+} & V_2|_{\mathcal{B}_0\cap\mathcal{B}_-}\end{bmatrix}}$ is the closure as linear relation (possibly multi-valued). Clearly, if (7.1) holds, then there is already equality.

Example 7.8. Let $(\mathcal{B}_+, \mathcal{B}_0, \mathcal{B}_-)$ be a quasi Gelfand triple that satisfies all conditions of Theorem 7.6 and let $M \in \mathcal{L}(\mathcal{B}_0)$ be coercive (i.e. $M \geq c\mathrm{I}$, $c > 0$). Then $V_1 := \mathrm{I}$, $V_2 := M$ fulfill all conditions of Theorem 7.6:

(i) Setting $S = M^{\frac{1}{2}}$ and $T = M^{-\frac{1}{2}}$ in Corollary 5.27 implies the closedness of $\begin{bmatrix} \mathrm{I}\big|_{\mathcal{B}_0\cap\mathcal{B}_+} & M\big|_{\mathcal{B}_0\cap\mathcal{B}_-}\end{bmatrix}$.

(ii) For $(x,y) \in \ker\begin{bmatrix} V_1 & V_2\end{bmatrix}$ we have $x = -My$. Since M is positive this yields

$$\operatorname{Re}\langle x,y\rangle_{\mathcal{B}_0} = \operatorname{Re}\langle -My, y\rangle = -\langle My, y\rangle \leq 0.$$

(iii) $V_1V_2^* + V_2V_1^* = M^* + M = 2\operatorname{Re} M \geq 0$.

Moreover, Corollary 5.27 also implies the surjectivity of $\begin{bmatrix} \mathrm{I}\big|_{\mathcal{B}_0\cap\mathcal{B}_+} & M\big|_{\mathcal{B}_0\cap\mathcal{B}_-}\end{bmatrix}$.

Actually, it would have been enough, if $M \in \mathcal{L}(\mathcal{B}_0)$ was boundedly invertible and accretive. Clearly, also $V_1 := M$, $V_2 := \mathrm{I}$ fulfill all conditions.

8. Port-hamiltonian systems as boundary control systems.

We will recall the notion of boundary control systems, scattering passive and impedance passive in the manner of [12]. We will show that a port-Hamiltonian system can be described as such a system. This concept already provides solution theory (see i.e. [11, Lemma 2.6]). It is well known that every scattering passive boundary control system induces a scattering passive well-posed linear system.

Definition 8.1. A *colligation* $\Xi := \left(\begin{bmatrix} G\\ L\\ K\end{bmatrix}; \begin{bmatrix}\mathcal{U}\\ \mathcal{X}\\ \mathcal{Y}\end{bmatrix}\right)$ consists of the three Hilbert spaces $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$, and the three linear maps G, L, and K, with the same domain $\mathcal{Z} \subseteq \mathcal{X}$ and with values in $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$, respectively.

Definition 8.2. A colligation $\Xi := \left(\begin{bmatrix} G\\ L\\ K\end{bmatrix}; \begin{bmatrix}\mathcal{U}\\ \mathcal{X}\\ \mathcal{Y}\end{bmatrix}\right)$ is an *(internally well-posed) boundary control system*, if

(i) the operator $\begin{bmatrix} G\\ L\\ K\end{bmatrix}$ is closed from $\mathcal{X}$ to $\begin{bmatrix}\mathcal{U}\\ \mathcal{X}\\ \mathcal{Y}\end{bmatrix}$,

(ii) the operator G is surjective, and

(iii) the operator $A := L\big|_{\ker G}$ generates a contraction semigroup on $\mathcal{X}$.

We think of the operators in this definition as determining a system via

$$\begin{aligned} u(t) &= Gx(t), \\ \dot{x}(t) &= Lx(t), \quad x(0) = x_0, \\ y(t) &= Kx(t). \end{aligned} \tag{8.1}$$

We call $\mathcal{U}$ the *input space*, $\mathcal{X}$ the *state space*, $\mathcal{Y}$ the *output space* and $\mathcal{Z}$ the *solution space*.

Definition 8.3. Let $\Xi = \left(\begin{bmatrix} G \\ L \\ K \end{bmatrix}; \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{Y} \end{bmatrix}\right)$ be a colligation. If Ξ is a boundary control system such that

$$2\operatorname{Re}\langle Lx, x\rangle_{\mathcal{X}} + \|Kx\|_{\mathcal{Y}}^2 \leq \|Gx\|_{\mathcal{U}}^2 \quad \text{for} \quad x \in \mathcal{Z}, \tag{8.2}$$

then it is *scattering passive* and it is *scattering energy preserving* if we have equality in (8.2).

We say Ξ is *impedance passive* (*energy preserving*), if $\mathcal{Y} = \mathcal{U}'$, $\Psi : \mathcal{U}' \to \mathcal{U}$ is the unitary identification mapping and $\tilde{\Xi} := \left(\begin{bmatrix} \frac{1}{\sqrt{2}}(G+\Psi K) \\ L \\ \frac{1}{\sqrt{2}}(G-\Psi K) \end{bmatrix}; \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{U} \end{bmatrix}\right)$ is scattering passive (energy preserving).

Note that an impedance passive (energy preserving) colligation Ξ does not need to be a boundary control system. If $\mathcal{U} = \mathcal{Y}$, then Ψ is the identity mapping.

Corresponding to a port-Hamiltonian system we want to introduce the following operators

$$\begin{aligned} G_{\mathrm{p}} &:= S_+ \begin{bmatrix} \bar{\pi}_L & 0 \end{bmatrix} \mathcal{H} \colon && \mathcal{H}^{-1}(H(P_\partial, \Omega)) \subseteq \mathcal{X}_{\mathcal{H}} \to S\mathcal{V}_L, \\ L_{\mathrm{p}} &:= (P_\partial + P_0)\mathcal{H} \colon && \mathcal{H}^{-1}(H(P_\partial, \Omega)) \subseteq \mathcal{X}_{\mathcal{H}} \to \mathcal{X}_{\mathcal{H}}, \\ K_{\mathrm{p}} &:= (S^*)_-^{-1} \begin{bmatrix} 0 & \bar{L}_\nu \end{bmatrix} \mathcal{H} \colon && \mathcal{H}^{-1}(H(P_\partial, \Omega)) \subseteq \mathcal{X}_{\mathcal{H}} \to (S\mathcal{V}_L)', \end{aligned}$$

where $S \in \mathcal{L}(L^2(\partial\Omega)^{m_1})$ is boundedly invertible, and S_+ and $(S^*)_-^{-1}$ denote their extension on $\mathcal{V}_L$ and $\mathcal{V}_L'$ respectively (see Corollary 5.26). By Lemma 5.28 also G_{p} and K_{p} establish a boundary triple for L_{p} restricted to $H_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega) \times H(L_\partial, \Omega)$ and $(S_+\mathcal{V}_{L,\Gamma_1}, SL_\pi^2(\Gamma_1), (S_+\mathcal{V}_{L,\Gamma_1})')$ is a quasi Gelfand triple For simplification S can be imagined to be the identity mapping. We still have Γ_0, Γ_1 as a splitting with thin boundaries of $\partial\Omega$.

Corollary 8.4. *The colligation* $\left(\begin{bmatrix} G_{\mathrm{p}} \\ L_{\mathrm{p}} \\ K_{\mathrm{p}} \end{bmatrix}; \begin{bmatrix} S_+\mathcal{V}_{L,\Gamma_1} \\ \mathcal{X}_{\mathcal{H}} \\ (S_+\mathcal{V}_{L,\Gamma_1})' \end{bmatrix}\right)$ *with solution space*

$$\mathcal{Z} = \mathcal{H}^{-1}\big(H_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega) \times H(L_\partial, \Omega)\big)$$

is a boundary control system.

Proof. Since L_{p} is closed on $\mathcal{X}_{\mathcal{H}}$ with domain $\mathcal{Z}$, and G_{p} and K_{p} are continuous with the graph norm of L_{p}, we have $\begin{bmatrix} G_{\mathrm{p}} & L_{\mathrm{p}} & K_{\mathrm{p}} \end{bmatrix}^{\mathsf{T}}$ is closed. By construction G_{p} with domain $\mathcal{Z}$ maps onto $S_+\mathcal{V}_{L,\Gamma_0}$. Since G_{p} is one operator of a boundary triple for L_{p}, the restriction $L_{\mathrm{p}}\big|_{\ker G_{\mathrm{p}}}$ is skew-adjoint and therefore a generator of a contraction semigroup. ❑

Proposition 8.5. *Let* $R \in \mathcal{L}(SL_\pi^2(\Gamma_1))$ *be coercive. Then the colligation* $\Xi = \left(\begin{bmatrix} \frac{1}{\sqrt{2}}(G_{\mathrm{p}}+RK_{\mathrm{p}}) \\ L_{\mathrm{p}} \\ \frac{1}{\sqrt{2}}(G_{\mathrm{p}}-RK_{\mathrm{p}}) \end{bmatrix}; \begin{bmatrix} \mathcal{U} \\ \mathcal{X}_{\mathcal{H}} \\ \mathcal{Y} \end{bmatrix}\right)$ *with* $\mathcal{U} = \mathcal{Y} = SL_\pi^2(\Gamma_1)$ *endowed with* $\|f\|_{\mathcal{U}} = \|f\|_{\mathcal{Y}} =$

$\|R^{-1/2} f\|_{L^2}$ *and solution space*

$$\mathcal{Z} = \{x \in \mathcal{H}^{-1}(H_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega) \times H(L_\partial, \Omega)) : G_{\mathrm{p}} x, K_{\mathrm{p}} x \in SL^2_\pi(\Gamma_1)\}.$$

is a scattering energy preserving boundary control system

Proof. Let $(x_n, [G_{\mathrm{p}} x_n \quad L_{\mathrm{p}} x_n \quad K_{\mathrm{p}} x_n]^{\mathsf{T}})_{n\in\mathbb{N}}$ be a sequence in $[G_{\mathrm{p}} \quad L_{\mathrm{p}} \quad K_{\mathrm{p}}]^{\mathsf{T}}$ (restricted to $\mathcal{Z}$) that converges to $(x, [f \quad y \quad g]^{\mathsf{T}}) \in \mathcal{X}_{\mathcal{H}} \times \mathcal{U} \times \mathcal{X}_{\mathcal{H}} \times \mathcal{U}$. Since L_{p} with domain $H(P_\partial, \Omega)$ is a closed operator and $H_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega) \times H(L_\partial, \Omega)$ is closed in $H(P_\partial, \Omega)$, we conclude that $x \in \mathcal{H}^{-1}(H_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega) \times H(L_\partial, \Omega))$ and $y = L_{\mathrm{p}} x$. Hence, $G_{\mathrm{p}} x_n$ converges in $S_+\mathcal{V}_{L,\Gamma_1}$ to $G_{\mathrm{p}} x$ and in $SL^2_\pi(\Gamma_1)$ to f. Since $(S_+\mathcal{V}_{L,\Gamma_1}, SL^2_\pi(\Gamma_1), (S_+\mathcal{V}_{L,\Gamma_1})')$ is a quasi Gelfand triple, we have $G_{\mathrm{p}} x = f$. Analogously, we conclude $K_{\mathrm{p}} x = g$. Therefore, $x \in \mathcal{Z}$ and $[G_{\mathrm{p}} \quad L_{\mathrm{p}} \quad K_{\mathrm{p}}]^{\mathsf{T}}$ is closed, which implies that also $\left[\frac{1}{\sqrt{2}}(G_{\mathrm{p}} + RK_{\mathrm{p}}) \quad L_{\mathrm{p}} \quad \frac{1}{\sqrt{2}}(G_{\mathrm{p}} - RK_{\mathrm{p}})\right]^{\mathsf{T}}$ is closed.

By Example 7.8 and Theorem 7.6 $L_{\mathrm{p}}\big|_{\ker \frac{1}{\sqrt{2}}(G_{\mathrm{p}}+RK_{\mathrm{p}})}$ generates a contraction semigroup.

The surjectivity of $\left[\begin{smallmatrix} G_{\mathrm{p}} \\ K_{\mathrm{p}} \end{smallmatrix}\right]$ and Example 7.8 gives the surjectivity of $\frac{1}{\sqrt{2}}(G_{\mathrm{p}}+RK_{\mathrm{p}})$. Since $(\mathcal{V}_L, G_{\mathrm{p}}, \Psi K_{\mathrm{p}})$ is a boundary triple for L_{p}, we have

$$\begin{aligned}
2\operatorname{Re}\langle L_{\mathrm{p}} x, x\rangle_{\mathcal{X}_{\mathcal{H}}} &= 2\operatorname{Re}\langle G_{\mathrm{p}} x, K_{\mathrm{p}} x\rangle_{\mathcal{V}_L, \mathcal{V}_L'} = 2\operatorname{Re}\langle G_{\mathrm{p}} x, K_{\mathrm{p}} x\rangle_{L^2_\pi(\Gamma_1)} \\
&= \frac{1}{2}\big(\langle R^{-1} G_{\mathrm{p}} x, G_{\mathrm{p}} x\rangle_{L^2} + 2\operatorname{Re}\langle G_{\mathrm{p}} x, K_{\mathrm{p}} x\rangle_{L^2} + \langle RK_{\mathrm{p}} x, K_{\mathrm{p}} x\rangle_{L^2}\big) \\
&\quad - \frac{1}{2}\big(\langle R^{-1} G_{\mathrm{p}} x, G_{\mathrm{p}} x\rangle_{L^2} - 2\operatorname{Re}\langle G_{\mathrm{p}} x, K_{\mathrm{p}} x\rangle_{L^2} + \langle RK_{\mathrm{p}} x, K_{\mathrm{p}} x\rangle_{L^2}\big) \\
&= \big\|\tfrac{1}{\sqrt{2}}(G_{\mathrm{p}} + RK_{\mathrm{p}})x\big\|^2_{\mathcal{U}} - \big\|\tfrac{1}{\sqrt{2}}(G_{\mathrm{p}} - RK_{\mathrm{p}})x\big\|^2_{\mathcal{Y}},
\end{aligned}$$

which makes Ξ scattering energy preserving. ❑

Remark 8.6. Clearly, the previous proposition holds also true for the operator triple $\left[\frac{1}{\sqrt{2}}(RK_{\mathrm{p}} + G_{\mathrm{p}}) \quad L_{\mathrm{p}} \quad \frac{1}{\sqrt{2}}(RK_{\mathrm{p}} - G_{\mathrm{p}})\right]^{\mathsf{T}}$ and for G_{p} and K_{p} being swapped. Moreover, replacing L_{p} by $L_{\mathrm{p}} + J$, where $J \in \mathcal{L}(\mathcal{X}_{\mathcal{H}})$ is dissipative, yields a scattering passive system.

Hence, the port-Hamiltonian system with input u and output y described by the equations

$$\begin{aligned}
\sqrt{2}u(t,\zeta) &= \pi_L\big(\mathcal{H}(\zeta)x(t,\zeta)\big)_{L^{\mathsf{H}}} + RL_\nu\big(\mathcal{H}(\zeta)x(t,\zeta)\big)_L, && t \in \mathbb{R}_+, \zeta \in \Gamma_1, \\
\frac{\partial}{\partial t}x(t,\zeta) &= \sum_{i=1}^n \frac{\partial}{\partial \zeta_i} P_i\big(\mathcal{H}(\zeta)x(t,\zeta)\big) + P_0\big(\mathcal{H}(\zeta)x(t,\zeta)\big), && t \in \mathbb{R}_+, \zeta \in \Omega, \\
\sqrt{2}y(t,\zeta) &= \pi_L\big(\mathcal{H}(\zeta)x(t,\zeta)\big)_{L^{\mathsf{H}}} - RL_\nu\big(\mathcal{H}(\zeta)x(t,\zeta)\big)_L, && t \in \mathbb{R}_+, \zeta \in \Gamma_1, \\
0 &= \pi_L\big(\mathcal{H}(\zeta)x(t,\zeta)\big)_{L^{\mathsf{H}}}, && t \in \mathbb{R}_+, \zeta \in \Gamma_0, \\
x(0,\zeta) &= x_0(\zeta), && \zeta \in \Omega,
\end{aligned} \tag{8.3}$$

is scattering passive and in particular well-posed, as the following corollary will clarify. The mappings π_L and L_ν are used a little bit sloppy. There is always a pointwise a.e. description for these mappings, but due to compact notation we use π_L and L_ν.

Corollary 8.7. *The system* (8.3) *can be interpreted as the scattering energy preserving boundary control system*

$$\left(\begin{bmatrix} \frac{1}{\sqrt{2}}(G_{\mathrm{p}}+RK_{\mathrm{p}}) \\ L_{\mathrm{p}} \\ \frac{1}{\sqrt{2}}(G_{\mathrm{p}}-RK_{\mathrm{p}}) \end{bmatrix}; \begin{bmatrix} \mathcal{U} \\ \mathcal{X}_{\mathcal{H}} \\ \mathcal{Y} \end{bmatrix}\right),$$

with the assumptions of Proposition 8.5 *and* $S = \mathrm{I}$. *Replacing* L_{p} *with* $L_{\mathrm{p}} + J$ *for a dissipative* $J \in \mathcal{L}(\mathcal{X}_{\mathcal{H}})$ *yields a scattering passive boundary control system.*

Corollary 8.8. *With the setting of Proposition* 8.5 *the colligation*

$$\left(\begin{bmatrix} G_{\mathrm{p}} \\ L_{\mathrm{p}} \\ K_{\mathrm{p}} \end{bmatrix}; \begin{bmatrix} SL^2_\pi(\Gamma_1) \\ \mathcal{X}_{\mathcal{H}} \\ SL^2_\pi(\Gamma_1) \end{bmatrix}\right)$$

with solution space

$$\mathcal{Z} = \{x \in \mathcal{H}^{-1}(H_{\Gamma_0}(L_\partial^{\mathsf{H}}, \Omega) \times H(L_\partial, \Omega)) : G_{\mathrm{p}}x, K_{\mathrm{p}}x \in SL^2_\pi(\Gamma_1)\}$$

is impedance energy preserving.

Proof. This is a direct consequence of Proposition 8.5 for $R = \mathrm{I}$. ❑

Note that the colligations in Corollary 8.4 and Corollary 8.8 are the same but the solution spaces are slightly different. The colligation in Corollary 8.8 is in general not necessarily a boundary control system.

Example 8.9 (Wave equation)**.** Let $\rho \in L^\infty(\Omega)$ be the mass density and $T \in L^\infty(\Omega)^{n\times n}$ be the Young modulus, such that $\frac{1}{\rho} \in L^\infty(\Omega)$, $T(\zeta)^{\mathsf{H}} = T(\zeta)$ and $T(\zeta) \geq \delta\mathrm{I}$ for a $\delta > 0$ and almost every $\zeta \in \Omega$. Then the wave equation

$$\frac{\partial^2}{\partial t^2} w(t,\xi) = \frac{1}{\rho(\xi)} \operatorname{div}\left(T(\xi) \operatorname{grad} w(t,\xi)\right),$$

can be formulated as a port-Hamiltonian system by choosing the state variable $x(t,\zeta) = \left[\begin{smallmatrix} \rho(\xi)\frac{\partial}{\partial t} w(t,\zeta) \\ \operatorname{grad} w(t,\zeta)\end{smallmatrix}\right]$. Then the PDE looks like

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{bmatrix}}_{=P_\partial} \underbrace{\begin{bmatrix} \frac{1}{\rho} & 0 \\ 0 & T \end{bmatrix}}_{=\mathcal{H}} x.$$

This is shown in section 3 of [9]. This is exactly the port-Hamiltonian system we get from choosing L as in Example 3.3. From Example 6.2 and Example 6.6 we know that the boundary operators are γ_0 and the extension of $\nu \cdot \gamma_0$. Therefore,

$$\begin{aligned} \sqrt{2}u(t,\zeta) &= \nu \cdot \left(T(\zeta) \operatorname{grad} w(t,\zeta)\right) + \frac{\partial}{\partial t} w(t,\zeta), && t \in \mathbb{R}_+, \zeta \in \Gamma_1, \\ \frac{\partial^2}{\partial t^2} w(t,\xi) &= \frac{1}{\rho(\xi)} \operatorname{div}\left(T(\xi) \operatorname{grad} w(t,\xi)\right), && t \in \mathbb{R}_+, \zeta \in \Omega, \\ \sqrt{2}y(t,\zeta) &= \nu \cdot \left(T(\zeta) \operatorname{grad} w(t,\zeta)\right) - \frac{\partial}{\partial t} w(t,\zeta), && t \in \mathbb{R}_+, \zeta \in \Gamma_1, \\ 0 &= \frac{\partial}{\partial t} w(t,\zeta), && t \in \mathbb{R}_+, \zeta \in \Gamma_0, \end{aligned}$$

can be modeled by a scattering passive and well-posed boundary control system, by Corollary 8.7.

Example 8.10 (Maxwell's equations)**.** Let $\Omega \subseteq \mathbb{R}^3$ be as in Assumption 3.1 and $L = (L_i)_{i=1}^3$ be as in Example 3.4. In this example we have already showed $L_\partial = \operatorname{rot}$ and $L_\nu f = \nu \times f$. The corresponding differential operator for the port-Hamiltonian PDE is

$$P_\partial = \begin{bmatrix} 0 & L_\partial \\ L_\partial^{\mathsf{H}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \operatorname{rot} \\ -\operatorname{rot} & 0 \end{bmatrix}.$$

We write the state as $x = \left[\begin{smallmatrix} \mathbf{D} \\ \mathbf{B} \end{smallmatrix}\right]$, where $\mathbf{D}, \mathbf{B} \in \mathbb{K}^3$. We also want to introduce the positive scalar functions ϵ, μ, g and r such that

$$\epsilon, \frac{1}{\epsilon}, \mu, \frac{1}{\mu}, g \in L^\infty(\Omega) \quad \text{and} \quad r, \frac{1}{r} \in L^\infty(\Gamma_1).$$

Furthermore, we define the Hamiltonian density by $\mathcal{H}(\zeta) := \begin{bmatrix} \frac{1}{\epsilon(\zeta)} & 0 \\ 0 & \frac{1}{\mu(\zeta)} \end{bmatrix}$, where each block is a 3×3 matrix. At last we define $\left[\begin{smallmatrix} \mathbf{E} \\ \mathbf{H} \end{smallmatrix}\right] := \mathcal{H} \left[\begin{smallmatrix} \mathbf{D} \\ \mathbf{B} \end{smallmatrix}\right]$, so that we have the same notation as in [16].

The projection on $\overline{\operatorname{ran} L_\nu}$ is given by $g \mapsto (\nu \times g) \times \nu$, therefore $\bar{\pi}_L$ is the extension of $g \mapsto (\nu \times \gamma_0 g) \times \nu$ to $H(L_\partial^{\mathsf{H}}, \Omega)$. The mapping π_τ from [16] can be compared with $\bar{\pi}_L$ but is not exactly the same, since they have different domains and codomains. We have $\pi_\tau \colon H^1(\Omega)^3 \to V_\tau \subseteq L^2(\partial\Omega)^3$ and $\bar{\pi}_L \colon H(\operatorname{rot}, \Omega) \to \mathcal{V}_L$ is its extension, if we change the norms in the domain and codomain of π_τ. However, $\mathcal{V}_L$ cannot be embedded into $L^2(\partial\Omega)^3$.

Note that by Example A.4 neither $\bar{\pi}_L$ nor $\bar{L}_\nu^{\Gamma_1}$ map even into $L^2_\pi(\Gamma_1)$, therefore it is really necessary to use a quasi Gelfand triple instead of an "ordinary" Gelfand triple.

The corresponding boundary control system is a model for Maxwell's equations in the following form

$$\begin{aligned}
\sqrt{2} u(t,\zeta) &= r(\zeta)\nu(\zeta) \times \mathbf{H}(t,\zeta) + (\nu(\zeta) \times \mathbf{E}(t,\zeta)) \times \nu(\zeta), && t \in \mathbb{R}_+, \zeta \in \Gamma_1, \\
\frac{\partial}{\partial t}\mathbf{D}(t,\zeta) &= \operatorname{rot} \mathbf{H}(t,\zeta) - g(\zeta)\mathbf{E}(t,\zeta), && t \in \mathbb{R}_+, \zeta \in \Omega, \\
\frac{\partial}{\partial t}\mathbf{B}(t,\zeta) &= -\operatorname{rot} \mathbf{E}(t,\zeta), && t \in \mathbb{R}_+, \zeta \in \Omega, \\
\sqrt{2} y(t,\zeta) &= r(\zeta)\nu(\zeta) \times \mathbf{H}(t,\zeta) - (\nu(\zeta) \times \mathbf{E}(t,\zeta)) \times \nu(\zeta), && t \in \mathbb{R}_+, \zeta \in \Gamma_1, \\
0 &= (\nu(\zeta) \times \mathbf{E}(t,\zeta)) \times \nu(\zeta), && t \in \mathbb{R}_+, \zeta \in \Gamma_0,
\end{aligned}$$

and is scattering passive by Corollary 8.7, where we set $J = \left[\begin{smallmatrix} -g & 0 \\ 0 & 0 \end{smallmatrix}\right] \mathcal{H}$.

Note that, following the trick in [16, Proposition 6.1], Gauß's law $\operatorname{div} \mathbf{D} = \rho$ is satisfied by simply defining ρ by this formula and Gauß's law for magnetism $\operatorname{div} \mathbf{B} = 0$ is automatically satisfied, if the initial condition satisfies it. This can be seen, if we apply div on both sides of $\frac{\partial}{\partial t}\mu\mathbf{H} = -\operatorname{rot} \mathbf{E}$ and noting that $\operatorname{div} \mu\mathbf{H} = \operatorname{div} \mathbf{B}$ is constant in time ($\operatorname{div} \operatorname{rot} = 0$). This has to be understood in the sense of distributions. However, for classical solutions this can also be understood in the classical sense.

Example 8.11 (Mindlin plate)**.** Let $\Omega \subseteq \mathbb{R}^2$ be as in Assumption 3.1. Let us consider the differential operator P_∂ and the skew-symmetric matrix P_0 given by

$$P_\partial \coloneqq \left[\begin{array}{ccc|ccccc} 0 & 0 & 0 & 0 & 0 & 0 & \partial_1 & \partial_2 \\ 0 & 0 & 0 & \partial_1 & 0 & \partial_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_2 & \partial_1 & 0 & 0 \\ \hline 0 & \partial_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial_2 & \partial_1 & 0 & 0 & 0 & 0 & 0 \\ \partial_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \partial_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], P_0 \coloneqq \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to derive the corresponding $P = (P_i)_{i=1}^2$ and $L = (L_i)_{i=1}^2$. We define a Hamiltonian density by

$$\mathcal{H} = \begin{bmatrix} \frac{1}{\rho h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12}{\rho h^3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{12}{\rho h^3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & & & 0 & 0 \\ 0 & 0 & 0 & & \boldsymbol{D}_b & & 0 & 0 \\ 0 & 0 & 0 & & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \multicolumn{2}{c}{\boldsymbol{D}_s} \\ 0 & 0 & 0 & 0 & 0 & 0 & & \end{bmatrix},$$

where ρ, h are strictly positive functions, $\boldsymbol{D}_b(\zeta)$ is a coercive 3×3 matrix and $\boldsymbol{D}_s(\zeta)$ is a coercive 2×2 matrix, such that all conditions on $\mathcal{H}$ in Definition 4.1 are satisfied. We write the state variable x as

$$\boldsymbol{\alpha} \coloneqq \begin{bmatrix} \rho h v & \rho \frac{h^3}{12} w_1 & \rho \frac{h^3}{12} w_2 & \kappa_{1,1} & \kappa_{2,2} & \kappa_{1,2} & \gamma_{1,3} & \gamma_{2,3} \end{bmatrix}^\mathsf{T},$$

where we stick to the notation in [2] except that we renamed the coordinates x, y and z as 1, 2 and 3. Furthermore, we have

$$\mathbf{e} \coloneqq \mathcal{H}\boldsymbol{\alpha} = \begin{bmatrix} v & w_1 & w_2 & M_{1,1} & M_{2,2} & M_{1,2} & Q_1 & Q_2 \end{bmatrix}^\mathsf{T}.$$

We don't want to go into details about the physical meaning of these state variables. We just want to make it easier to translate the results into the notation of [2]. So the port-Hamiltonian PDE

$$\frac{\partial}{\partial t} x = (P_\partial + P_0)\mathcal{H}x \quad \text{looks like} \quad \frac{\partial}{\partial t}\boldsymbol{\alpha} = (P_\partial + P_0)\mathbf{e}.$$

The corresponding boundary operator is

$$L_\nu f = \begin{bmatrix} 0 & 0 & 0 & \nu_1 & \nu_2 \\ \nu_1 & 0 & \nu_2 & 0 & 0 \\ 0 & \nu_2 & \nu_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = \begin{bmatrix} \nu \cdot \begin{bmatrix} f_4 \\ f_5 \end{bmatrix} \\ \nu \cdot \begin{bmatrix} f_1 \\ f_3 \end{bmatrix} \\ \nu \cdot \begin{bmatrix} f_3 \\ f_2 \end{bmatrix} \end{bmatrix}.$$

Since $\|\nu(\zeta)\| = 1$, at least $\nu_1(\zeta) \neq 0$ or $\nu_2(\zeta) \neq 0$. This can be used to show that $\operatorname{ran} L_\nu = L^2(\partial\Omega)^3$. Therefore, $\bar{\pi}_L$ is the extension of the boundary trace operator γ_0 to $H(L_\partial^\mathsf{H}, \Omega)$.

Since there is no direct physical meaning to the boundary variables

$$\begin{bmatrix}0 & L_\nu\end{bmatrix}\mathbf{e} = \begin{bmatrix}\nu\cdot\left[\begin{smallmatrix}Q_1\\Q_2\end{smallmatrix}\right]\\ \nu\cdot\left[\begin{smallmatrix}M_{1,1}\\M_{1,2}\end{smallmatrix}\right]\\ \nu\cdot\left[\begin{smallmatrix}M_{1,2}\\M_{2,2}\end{smallmatrix}\right]\end{bmatrix} \quad\text{and}\quad \begin{bmatrix}\pi_L & 0\end{bmatrix}\mathbf{e} = \begin{bmatrix}v\\w_1\\w_2\end{bmatrix},$$

we define $\eta \coloneqq \left[\begin{smallmatrix}-\nu_2\\ \nu_1\end{smallmatrix}\right]$ and apply the unitary transformation $S = \left[\begin{smallmatrix}1 & 0 & 0\\ 0 & \nu_1 & \nu_2\\ 0 & -\nu_2 & \nu_1\end{smallmatrix}\right]$ to obtain

$$\begin{bmatrix}Q_\nu\\M_{\nu,\nu}\\M_{\nu,\eta}\end{bmatrix} \coloneqq S\begin{bmatrix}\nu\cdot\left[\begin{smallmatrix}Q_1\\Q_2\end{smallmatrix}\right]\\ \nu\cdot\left[\begin{smallmatrix}M_{1,1}\\M_{1,2}\end{smallmatrix}\right]\\ \nu\cdot\left[\begin{smallmatrix}M_{1,2}\\M_{2,2}\end{smallmatrix}\right]\end{bmatrix} \quad\text{and}\quad \begin{bmatrix}v\\w_\nu\\w_\eta\end{bmatrix} \coloneqq \underbrace{(S^*)^{-1}}_{=S}\begin{bmatrix}v\\w_1\\w_2\end{bmatrix},$$

which have a physical interpretation; see [2]. Hence, by Corollary 8.8 the system

$$\begin{aligned}
u &= \begin{bmatrix}Q_\nu & M_{\nu,\nu} & M_{\nu,\eta}\end{bmatrix}^\top, && \text{on } \mathbb{R}_+\times\Gamma_1,\\
\frac{\partial}{\partial t}\boldsymbol{\alpha} &= (P_\partial + P_0)\mathbf{e}, && \text{on } \mathbb{R}_+\times\Omega,\\
y &= \begin{bmatrix}v & w_\nu & w_\eta\end{bmatrix}^\top, && \text{on } \mathbb{R}_+\times\Gamma_1,\\
0 &= \begin{bmatrix}v & w_\nu & w_\eta\end{bmatrix}^\top, && \text{on } \mathbb{R}_+\times\Gamma_0,
\end{aligned}$$

for the Mindlin plate is impedance energy preserving, which is exactly the system in [2].

Appendix A. Counter examples and technical lemmas. The next example shows that it is possible to have item (i) and item (ii) of a "boundary triple" for an operator A (Definition 2.1) without A being the adjoint of a skew-symmetric operator. Moreover, it shows that in this situation Lemma 2.2 does not hold. This demonstrates the importance of A being the adjoint of a skew-symmetric operator in the definition.

Example A.1. Let $A = \left[\begin{smallmatrix}0 & \frac{\mathrm{d}}{\mathrm{d}\xi}\\ \frac{\mathrm{d}}{\mathrm{d}\xi} & 0\end{smallmatrix}\right]$ be an operator on $L^2(0,1)^2$ with $\operatorname{dom} A = H^1(0,1)^2$. By Remark 3.7 the operator A is the adjoint of a skew-symmetric operator. Integration by parts yields

$$\begin{aligned}
\langle Af, g\rangle + \langle f, Ag\rangle &= \int_0^1 \left\langle \begin{bmatrix}f_2'\\f_1'\end{bmatrix}, \begin{bmatrix}g_1\\g_2\end{bmatrix}\right\rangle \mathrm{d}\xi + \int_0^1 \left\langle \begin{bmatrix}f_1\\f_2\end{bmatrix}, \begin{bmatrix}g_2'\\g_1'\end{bmatrix}\right\rangle \mathrm{d}\xi\\
&= \int_0^1 (f_2'g_1 + f_1'g_2 + f_1g_2' + f_2g_1')\,\mathrm{d}\xi = f_2g_1\Big|_0^1 + f_1g_2\Big|_0^1\\
&= f_2(1)g_1(1) - f_2(0)g_1(0) + f_1(1)g_2(1) - f_1(0)g_2(0)\\
&= \Big\langle \underbrace{\begin{bmatrix}f_2(1)\\-f_2(0)\end{bmatrix}}_{B_2f}, \underbrace{\begin{bmatrix}g_1(1)\\g_1(0)\end{bmatrix}}_{B_1g}\Big\rangle + \Big\langle \underbrace{\begin{bmatrix}f_1(1)\\f_1(0)\end{bmatrix}}_{B_1f}, \underbrace{\begin{bmatrix}g_2(1)\\-g_2(0)\end{bmatrix}}_{B_2g}\Big\rangle.
\end{aligned}$$

Defining $B_1 f \coloneqq \left[\begin{smallmatrix}f_1(1)\\f_1(0)\end{smallmatrix}\right]$ and $B_2 f \coloneqq \left[\begin{smallmatrix}f_2(1)\\-f_2(0)\end{smallmatrix}\right]$ yields

$$\langle Af, g\rangle + \langle f, Ag\rangle = \langle B_1 f, B_2 g\rangle + \langle B_2 f, B_1 g\rangle. \tag{A.1}$$

The mapping $\left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right] : \operatorname{dom} A \to \mathbb{R}^4$ is surjective (this can be seen by choosing f_1 and f_2 to be linear interpolations). So $(\mathbb{R}^2, B_1, B_2)$ is a boundary triple for A.

We define $\hat{A}$ as the restriction of A on $H^1_{\{1\}=0}(0,1) \times H^1_{\{0\}=\{1\}}(0,1)$, where

$$\begin{aligned} H^1_{\{1\}=0}(0,1) &\coloneqq \{f \in H^1(0,1) : f(1) = 0\}, \quad \text{and} \\ H^1_{\{0\}=\{1\}}(0,1) &\coloneqq \{f \in H^1(0,1) : f(0) = f(1)\}. \end{aligned}$$

Therefore, we can reformulate (A.1) for $f, g \in \operatorname{dom} \hat{A}$

$$\langle \hat{A}f, g\rangle + \langle f, \hat{A}g\rangle = -f_1(0)g_2(0) + f_2(0)(-g_1(0))$$

By defining $F_1 f \coloneqq -f_1(0)$ and $F_2 f \coloneqq f_2(0)$ we again have that $\left[\begin{smallmatrix} F_1 \\ F_2 \end{smallmatrix}\right] : \operatorname{dom} \hat{A} \to \mathbb{R}^2$ is surjective. However $\hat{A}$ is not the adjoint of a skew-symmetric operator. If it were, then $(\mathbb{R}^2, F_1, F_2)$ would be a boundary triple for $\hat{A}$ and

$$\hat{A}^* = -\hat{A}\big|_{\ker F_1 \cap \ker F_2} = -A\big|_{H^1_0(0,1)^2} = A^*.$$

which is not true since $\hat{A}$ is certainly not dense in A. In fact, with the boundary triple for A we get that the adjoint of $\hat{A}$ is $-A\big|_{H^1_{\{0\}=\{1\}}(0,1) \times H^1_{\{0\}=0}(0,1)}$.

Lemma A.2. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a normed vector space X that converges w.r.t. the weak-$*$ topology to an $x_0 \in X$. Then $(x_n)_{n\in\mathbb{N}}$ is bounded i.e. $\sup_{n\in\mathbb{N}} \|x_n\|_X < +\infty$.*

Proof. Let ι denote the canonical embedding from X into X'' that maps x to $\langle x, .\rangle_{X,X'}$. Then, by assumption, for every fixed $\phi \in X'$ $(\iota x_n)(\phi) \to (\iota x_0)(\phi)$, in particular $\sup_{n\in\mathbb{N}} |(\iota x_n)(\phi)| < \infty$. The principle of uniform boundedness yields $\sup_{n\in\mathbb{N}} \|\iota x_n\|_{X''} < +\infty$. Since $\|\iota x\|_{X''} = \|x\|_X$ for every $x \in X$, this proves the assertion. ❑

Lemma A.3. *Let $(x_n)_{n\in\mathbb{N}}$ be a weak convergent sequence in a Hilbert space H with limit x. Then there exists a subsequence $(x_{n(k)})_{k\in\mathbb{N}}$ such that*

$$\left\| \frac{1}{N} \sum_{k=1}^{N} x_{n(k)} - x \right\| \to 0.$$

Proof. We assume that $x = 0$. For the general result we just need to replace x_n by $x_n - x$.

We define the subsequence inductively: $n(1) = 1$ and for $k > 1$ we choose $n(k)$ such that

$$|\langle x_{n(k)}, x_{n(j)}\rangle| \leq \frac{1}{k} \quad \text{for all} \quad j < k.$$

This is possible, because $(x_n)_{n\in\mathbb{N}}$ converges weakly to 0. Note that in Hilbert spaces the weak topology and the weak-$*$ topology are the same. Hence, by Lemma A.2

$\sup_{n\in\mathbb{N}}\|x_n\| \leq C$. This yields

$$\begin{aligned}\left\|\frac{1}{N}\sum_{k=1}^{N} x_{n(k)}\right\|^2 &= \frac{1}{N^2}\sum_{k=1}^{N}\sum_{j=1}^{N}\langle x_{n(k)}, x_{n(j)}\rangle \\ &= \frac{1}{N^2}\sum_{k=1}^{N}\|x_{n(k)}\|^2 + \frac{1}{N^2}\sum_{j=1}^{N}\sum_{k=j+1}^{N} 2\operatorname{Re}\langle x_{n(k)}, x_{n(j)}\rangle \\ &\leq \frac{1}{N}C^2 + \frac{2}{N^2}\sum_{j=1}^{N}\sum_{k=j+1}^{N}\frac{1}{k} \leq \frac{C^2}{N} + \frac{1}{N}\ln(N) \to 0.\end{aligned}$$ ❑

Example A.4. Let $\Omega = (0,1)^3$ and $F\colon \Omega \to \mathbb{R}$ be defined by

$$F(x) = \frac{1}{\|x\|_2^{2/5}} = (x_1^2 + x_2^2 + x_3^2)^{-2/10}.$$

Then we define $f = \operatorname{grad} F$, which is

$$f(x) = \begin{bmatrix} -\frac{4}{10}x_1(x_1^2 + x_2^2 + x_3^2)^{-6/5} \\ -\frac{4}{10}x_2(x_1^2 + x_2^2 + x_3^2)^{-6/5} \\ -\frac{4}{10}x_3(x_1^2 + x_2^2 + x_3^2)^{-6/5} \end{bmatrix}.$$

Hence, $\operatorname{rot} f = \operatorname{rot}\operatorname{grad} F = 0$. We will show that f is in $L^2(\Omega)^3$:

$$\begin{aligned}\int_\Omega \|f(x)\|_2^2 \,\mathrm{d}x &= \int_\Omega \sum_{i=1}^{3}\frac{16}{100}x_i^2(x_1^2+x_2^2+x_3^2)^{-12/5}\,\mathrm{d}x = \frac{16}{100}\int_\Omega (x_1^2+x_2^2+x_3^2)^{-7/5}\,\mathrm{d}x \\ &\leq \int_{B_{\sqrt{3}}(0)} (x_1^2+x_2^2+x_3^2)^{-7/5}\,\mathrm{d}x = 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\sqrt{3}} r^{-14/5}r^2\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta \\ &= 4\pi\int_0^{\sqrt{3}} r^{-4/5}\,\mathrm{d}r = 4\pi 5 r^{1/5}\Big|_0^{\sqrt{3}} < +\infty.\end{aligned}$$

Therefore, f is even in $H(\operatorname{rot},\Omega)$. Let ν denote the normal vector on $\partial\Omega$. Then we show that $\nu \times f\big|_{\partial\Omega}$ is not in $L^2(\partial\Omega)^3$: Note that $\nu(\zeta) = \left[\begin{smallmatrix}0\\0\\-1\end{smallmatrix}\right]$ on $[0,1]\times[0,1]\times\{0\}$. Therefore,

$$\nu(\zeta)\times f(\zeta) = \begin{bmatrix} -\frac{4}{10}\zeta_2(\zeta_1^2+\zeta_2^2)^{-6/5} \\ \frac{4}{10}\zeta_1(\zeta_1^2+\zeta_2^2)^{-6/5} \\ 0 \end{bmatrix} \quad \text{for} \quad \zeta \in [0,1]\times[0,1]\times\{0\}$$

and consequently

$$\begin{aligned}\int_{\partial\Omega}\|\nu(\zeta)\times f(\zeta)\|_2^2\,\mathrm{d}\zeta &\geq \int_{[0,1]\times[0,1]\times\{0\}}\|\nu(\zeta)\times f(\zeta)\|_2^2\,\mathrm{d}\zeta \\ &= \frac{16}{100}\int_{[0,1]\times[0,1]}(\xi_1^2+\xi_2^2)^{-7/5}\,\mathrm{d}\xi.\end{aligned}$$

Since $[0,1]\times[0,1]$ contains the circular sector with arc $\frac{\pi}{2}$ and radius 1, we further have (by applying polar coordinates)

$$\begin{aligned}&\geq \frac{16}{100}\frac{\pi}{2}\int_0^1 r^{-14/5} r \,\mathrm{d}r = \frac{16}{100}\frac{\pi}{2}\int_0^1 r^{-9/5}\,\mathrm{d}r\\ &= -\frac{16}{100}\frac{\pi}{2}\frac{5}{4}r^{-4/5}\Big|_0^1 = +\infty.\end{aligned}$$

Hence, $f\in H(\mathrm{rot},\Omega)$, but $\nu\times f\big|_{\partial\Omega}\notin L^2(\partial\Omega)^3$. Since

$$(\nu(\zeta)\times f(\zeta))\times\nu(\zeta) = \begin{bmatrix} -\frac{4}{10}\zeta_1(\zeta_1^2+\zeta_2^2)^{-6/5}\\ -\frac{4}{10}\zeta_2(\zeta_1^2+\zeta_2^2)^{-6/5}\\ 0\end{bmatrix} \quad \text{for} \quad \zeta\in[0,1]\times[0,1]\times\{0\},$$

we also have $\left(\nu\times f\big|_{\partial\Omega}\right)\times\nu\notin L^2(\partial\Omega)^3$.

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