CHARACTERIZATIONS OF THE SOBOLEV SPACE H^1 ON THE BOUNDARY OF A STRONGLY LIPSCHITZ DOMAIN IN 3-D

NATHANAEL SKREPEK

ABSTRACT. In this work we investigate the Sobolev space $\mathrm{H}^1(\partial\Omega)$ on a strongly Lipschitz boundary $\partial\Omega$, i.e., Ω is a strongly Lipschitz domain (not necessarily bounded). In most of the literature this space is defined via charts and Sobolev spaces on flat domains. We show that there is a different approach via differential operators on Ω and a weak formulation directly on the boundary that leads to the same space. This second characterization of $\mathrm{H}^1(\partial\Omega)$ is in particular of advantage, when it comes to traces of $\mathrm{H}(\mathrm{curl}, \Omega)$ vector fields.

1. INTRODUCTION

We will give two characterizations of $\mathrm{H}^1(\partial\Omega)$, where Ω is a strongly Lipschitz domain (not necessarily bounded). The first is given via charts, which is the usual approach in literature, and the second is a weak characterization directly on the boundary, which is related to the weak characterization of an $\mathrm{L}^2(\partial\Omega)$ tangential trace for $\mathrm{H}(\mathrm{curl}, \Omega)$ fields.

Our main motivation is that the result we present serves us to fill details in [Cos90, Proof of Thm. 2], [BBBCD97, Section Le cas tridimensionnel], [BCS02, Proof of Thm. 5.1] and [Mon03, Proof of Lem. 3.53], where it is used. Unfortunately, without an explanation or a reference for its validity. Hence, we decided to address this issue.

In particular, if we regard an $f \in \mathrm{H}^1(\Omega)$, then $\nabla f \in \mathrm{H}(\mathrm{curl}, \Omega)$ follows automatically (because $\mathrm{curl}\,\nabla f = 0$). Every element of $\mathrm{H}(\mathrm{curl}, \Omega)$ possesses a tangential trace in an abstract boundary space and therefore also ∇f possesses a tangential trace. For smooth functions the tangential trace is well defined as an element of $\mathrm{L}^2(\partial\Omega)^3$. Moreover, for a smooth function the tangential trace of its gradient field coincides with the boundary gradient of its restriction to the boundary, see Lemma 3.4. This suggests the following claim.

Claim A. Let $f \in \mathrm{H}^1(\Omega)$. If the tangential trace of ∇f belongs to $\mathrm{L}^2(\partial\Omega)^3$, then $f\big|_{\partial\Omega}$ belongs to $\mathrm{H}^1(\partial\Omega)$.

However, there are two approaches to define "*the tangential trace belongs to* $\mathrm{L}^2(\partial\Omega)^3$": The strong approach via limits of smooth functions and the weak approach via a representation by an $\mathrm{L}^2(\partial\Omega)$ inner product. For the strong approach it is not hard to show that Claim A is true. However, it is more relevant to answer the question for the weak approach. Hence, we regard the claim with the weak characterization of L^2 tangential traces (Definition 4.1).

In fact both [BBBCD97] and [Mon03] are using Claim A (with weak L^2 tangential traces) to prove that both approaches (strong and weak) to L^2 tangential traces lead to the same objects, i.e., weak = strong. Hence, in order to avoid a circular

---

Date: February 6, 2024.

2020 *Mathematics Subject Classification.* 46E35, 46E36, 47F99.

Key words and phrases. Sobolev spaces, Lipschitz domains, Lipschitz boundaries, tangential traces, tangential gradients.

E-mail: nathanael.skrepek@math.tu-freiberg.de.

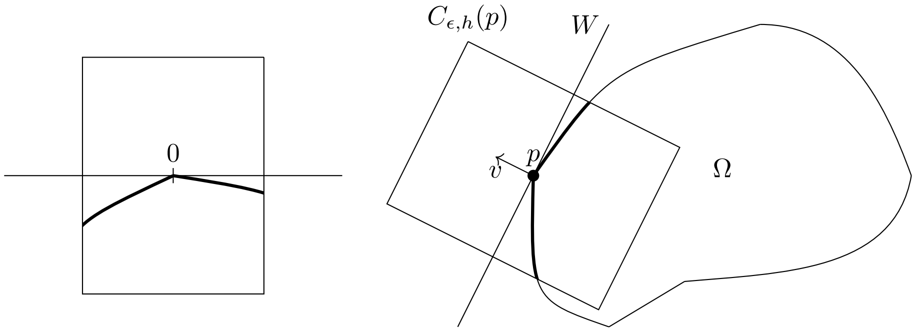

FIGURE 1. Lipschitz boundary

argument we have to resist the temptation to prove Claim A for strong L^2 tangential traces and conclude it for weak by "weak = strong".

In order to avoid the introduction of unnecessarily many concepts, we broke down the question to its core, which is an alternative approach to $\mathrm{H}^1(\partial\Omega)$, see Definition 3.6. Hence, we do not need the space $\mathrm{H}(\mathrm{curl},\Omega)$ and the abstract tangential trace at all, although these notions are the origin of the question. Nevertheless, in Section 4 we come back to the original question and show that Claim A holds true.

2. Strongly Lipschitz boundaries

Recall the definition of a strongly Lipschitz domain, see, e.g., [Gri85].

Definition 2.1. Let Ω be an open subset of $\mathbb{R}^d$. We say Ω is a *strongly Lipschitz domain*, if for every $p \in \partial\Omega$ there exist $\epsilon, h > 0$, a hyperplane $W = \operatorname{span}\{w_1, \dots, w_{d-1}\}$, where $\{w_1, \dots, w_{d-1}\}$ is an orthonormal basis of W, and a Lipschitz continuous function $a\colon (p + W) \cap \mathrm{B}_\epsilon(p) \to (-\frac{h}{2}, \frac{h}{2})$ such that

$$\begin{aligned} \partial\Omega \cap C_{\epsilon,h}(p) &= \{x + a(x)v \mid x \in (p+W) \cap \mathrm{B}_\epsilon(p)\}, \\ \Omega \cap C_{\epsilon,h}(p) &= \{x + sv \mid x \in (p+W) \cap \mathrm{B}_\epsilon(p), -h < s < a(x)\}, \end{aligned}$$

where v is the normal vector of W and $C_{\epsilon,h}(p)$ is the cylinder $\{x + \delta v \mid x \in (p + W) \cap \mathrm{B}_\epsilon(p), \delta \in (-h, h)\}$.

The boundary $\partial\Omega$ is then called *strongly Lipschitz boundary.*

Note that the condition $|a| < \frac{h}{2}$ is not really necessary, however it reduces technical constructions. If it was not already satisfied, we can force it by shrinking ϵ.

Locally the boundary is given by the graph of a Lipschitz function, see Figure 1. Therefore, we can define Lipschitz charts on $\partial\Omega$ in the following way. Let p, $C_{\epsilon,h}(p)$, W, v, a be as in Definition 2.1. We will also denote the matrix that contains the orthonormal basis of W as columns by W, i.e., $W \in \mathbb{R}^{d\times(d-1)}$. Hence, the mapping $\zeta \mapsto W^\mathsf{T}\zeta$ gives the coordinates (w.r.t. the basis $w_1, \dots, w_{d-1}$) of the orthogonal projection of ζ on the hyperplane W. We introduce a *strongly Lipschitz chart* locally at p by

$$k\colon \left\{ \begin{array}{rcl} \partial\Omega \cap C_{\epsilon,h}(p) & \to & \mathrm{B}_\epsilon(0) \subseteq \mathbb{R}^{d-1}, \\ \zeta & \mapsto & W^\mathsf{T}(\zeta - p). \end{array} \right.$$

We say that $\Gamma \coloneqq \partial\Omega \cap C_{\epsilon,h}(p)$ is the *chart domain* of k. Also every restriction of a chart to an open non-empty $\hat{\Gamma} \subseteq \Gamma$ (w.r.t. the trace topology) is again a chart with chart domain $\hat{\Gamma}$. The corresponding inverse chart is given by

$$k^{-1} \colon \begin{cases} \mathrm{B}_\epsilon(0) \subseteq \mathbb{R}^{d-1} & \to & \partial\Omega \cap C_{\epsilon,h}(p), \\ x & \mapsto & p + \sum_{i=1}^{d-1} x_i w_i + a(p + \sum_{i=1}^{d-1} x_i w_i) v. \end{cases}$$

In the case where k is a "restricted" chart, we have $k^{-1} \colon U \to \hat{\Gamma}$, where U is an open non-empty subset of $\mathrm{B}_\epsilon(0)$ in $\mathbb{R}^{d-1}$. For notational simplicity we just write $a(x)$ instead of $a\big(p + \sum_{i=1}^{d-1} x_i w_i\big)$. By this convention we have $a \colon U \subseteq \mathbb{R}^{d-1} \to \mathbb{R}$.

Note that in fact W, v and p establish an alternative coordinate system with origin p. Hence, by translation and rotation we can, most of the time, assume (w.l.o.g.) that $W = (e_1, \ldots, e_{d-1})$, $v = e_d$ and $p = 0$. This will also better transport the essence of our ideas. In this coordinate system we have

$$k\left(\begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_d \end{bmatrix}\right) = \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_{d-1} \end{bmatrix} \quad \text{and} \quad k^{-1}(x) = \begin{bmatrix} x \\ a(x) \end{bmatrix}.$$

However, sometimes it is not entirely obvious that we can reduce the general setting to this situation or the justification that such a reduction is valid is as difficult as working in the general setting in the first place. Hence, for completeness we will repeat the tricky parts for the general setting in the appendix.

Note that k^{-1} is Lipschitz continuous—since a is Lipschitz continuous by assumption—and therefore k^{-1} is a.e. differentiable by Rademacher's theorem, see, e.g., [AFP00, Thm. 2.14]. In particular, $k^{-1} \in \mathrm{W}^{1,\infty}(U)$ and therefore $\mathrm{d}k^{-1}$ is a bounded multiplication operator on $\mathrm{L}^2(U)$. Hence, if we don't write arguments (of functions), then we regard the functions as L^p objects and omit the comment "a.e.".

Let $k \colon \Gamma \to U$ be a strongly Lipschitz chart. The surface measure on $\partial\Omega$ is locally given by

$$\mu(\Upsilon) = \int_{k(\Upsilon)} \sqrt{\det(\mathrm{d}k^{-1})^\mathsf{T} \mathrm{d}k^{-1}} \,\mathrm{d}\lambda_{d-1} \quad \text{for} \quad \Upsilon \subseteq \Gamma,$$

where λ_{d-1} is the Lebesgue measure in $\mathbb{R}^{d-1}$. The surface measure is then defined by a partition of $\partial\Omega$. By Lindelöf's lemma there exists a countable partition, see, e.g., [Nag85, Ch. 3 § 4]. If $\partial\Omega$ is bounded then there exists even a finite partition. The surface measure is independent of the partition and the charts, see Proposition B.4. Hence, we can switch between the inner products of $\mathrm{L}^2(\Gamma)$ and $\mathrm{L}^2(U)$ by

$$\langle f, g \rangle_{\mathrm{L}^2(\Gamma)} = \Big\langle f \circ k^{-1}, \sqrt{\det(\mathrm{d}k^{-1})^\mathsf{T} \mathrm{d}k^{-1}}\, g \circ k^{-1} \Big\rangle_{\mathrm{L}^2(U)}.$$

Note, if $\partial\Omega$ is unbounded and f is integrable, then (by monotone convergence)

$$\lim_{r\to\infty} \int_{\partial\Omega \cap \mathrm{B}_r(0)} f \,\mathrm{d}\mu = \int_{\partial\Omega} f \,\mathrm{d}\mu,$$

where $\mathrm{B}_r(0)$ is the ball in $\mathbb{R}^d$. In particular for every $\epsilon > 0$ there exists an $r > 0$ such that $\big|\int_{\partial\Omega \setminus \mathrm{B}_r(0)} f \,\mathrm{d}\mu\big| \leq \epsilon$.

3. Preparation and main result

We will use for spaces with homogeneous boundary conditions the same notation as in [BPS16]: For an open set $M \subseteq \mathbb{R}^d$ we denote the set of C^∞ functions with compact support in M by

$$\mathring{\mathrm{C}}^\infty(M) := \{\Phi \in \mathrm{C}^\infty(\mathbb{R}^d) \,|\, \operatorname{supp}\Phi \subseteq M \text{ and } \operatorname{supp}\Phi \text{ is compact}\}.$$

Moreover, we denote the standard $\mathrm{L}^2(M)$ first order Sobolev space by $\mathrm{H}^1(M)$ and

$$\mathring{\mathrm{H}}^1(M) := \overline{\mathring{\mathrm{C}}^\infty(M)}^{\mathrm{H}^1(M)}.$$

The circle on top of $\mathrm{H}^1(M)$ indicates homogeneous boundary conditions.

In the following we assume $\Omega \subseteq \mathbb{R}^3$ to be a strongly Lipschitz domain. Moreover, we will assume that the strongly Lipschitz charts $k\colon \Gamma \subseteq \partial\Omega \to U$ are of the following form

$$k^{-1}\colon \begin{cases} U & \to & \Gamma, \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & \mapsto & \begin{bmatrix} x_1 \\ x_2 \\ a(x_1, x_2) \end{bmatrix}, \end{cases}$$

where U is an open subset of $\mathbb{R}^2$ and $a\colon U \to \mathbb{R}$ is a Lipschitz continuous mapping. The outward pointing normalized normal vector (as an element of L^∞) is then locally given by

$$\nu \circ k^{-1} = \frac{1}{\sqrt{1 + \|\nabla a\|^2}} \begin{bmatrix} -\partial_1 a \\ -\partial_2 a \\ 1 \end{bmatrix}.$$

In Appendix A we show, which modifications have to be done when we work with "general" strongly Lipschitz charts. We could also do everything for "general" strongly Lipschitz charts in the first place, however it does not transport the underlying ideas that well. Also we did not want to just say that we can always reduce everything to these "special" strongly Lipschitz charts, as sometimes it is not obvious how this "w.l.o.g." is justified.

Lemma 3.1. *Let $k\colon \Gamma \to U$ be a strongly Lipschitz chart. Then*

$$\det\big((\mathrm{d}k^{-1})^\mathsf{T} \mathrm{d}k^{-1}\big) = 1 + \|\nabla a\|^2.$$

Proof. Note that

$$\mathrm{d}k^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \partial_1 a & \partial_2 a \end{bmatrix} \quad \text{and} \quad (\mathrm{d}k^{-1})^\mathsf{T} \mathrm{d}k^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \partial_1 a \\ \partial_2 a \end{bmatrix} \begin{bmatrix} \partial_1 a & \partial_2 a \end{bmatrix}.$$

Hence, Lemma C.1 applied to $v = \left[\begin{smallmatrix} \partial_1 a(x) \\ \partial_2 a(x) \end{smallmatrix}\right]$ for a.e. $x \in U$ implies the claim. ❑

Recall the Moore-Penrose inverse: For an injective matrix A it is given by $A^\dagger = (A^\mathsf{T} A)^{-1} A^\mathsf{T}$. Our first approach to the first order Sobolev space on $\partial\Omega$ is well-known, see, e.g., [BCS02, beginning of Sec. 3], [Gri85, Def. 1.3.3.2] or [Neč12, after Ch. 2, Thm. 4.10].

Definition 3.2. Let Ω be additionally bounded and $f \in \mathrm{L}^2(\partial\Omega)$. We say $f \in \mathrm{H}^1(\partial\Omega)$, if for every strongly Lipschitz chart $k\colon \Gamma \to U$ we have $f \circ k^{-1} \in \mathrm{H}^1(U)$. The tangential gradient is then defined by

$$(\nabla_\tau f)\big|_\Gamma = \big[\mathrm{d}(f \circ k^{-1})(\mathrm{d}k^{-1})^\dagger\big]^\mathsf{T} \circ k = \big[(\mathrm{d}k^{-1})^{\dagger\,\mathsf{T}} \nabla_{\mathbb{R}^2}(f \circ k^{-1})\big] \circ k.$$

We endow $\mathrm{H}^1(\partial\Omega)$ with the following norm

$$\|f\|_{\mathrm{H}^1(\partial\Omega)} = \sqrt{\|f\|^2_{\mathrm{L}^2(\partial\Omega)} + \|\nabla_\tau f\|^2_{\mathrm{L}^2(\partial\Omega)}}.$$

Note that, if the previous definition is true for a set of charts whose chart domains cover $\partial\Omega$, then it is already true for all charts. Moreover, the definition of the tangential gradient is independent of the chart, see Proposition B.3.

Furthermore, note that the previous definition is fine, if we regard bounded domains Ω or their complements (finitely many charts cover the entire boundary). However, if we deal with domains Ω with unbounded boundaries (the boundary cannot be covered by finitely many charts), then local integrability does not lead to global integrability. Hence, we need to add an extra assumption to the definition of $\mathrm{H}^1(\partial\Omega)$.

Definition 3.3. Let $f \in \mathrm{L}^2_{\mathrm{loc}}(\partial\Omega)$. We say $f \in \mathrm{H}^1_{\mathrm{loc}}(\partial\Omega)$, if for every strongly Lipschitz chart $k\colon \Gamma \to U$ we have $f \circ k^{-1} \in \mathrm{H}^1(U)$. The tangential gradient is then defined by

$$(\nabla_\tau f)\big|_\Gamma = \big[\mathrm{d}(f \circ k^{-1})(\mathrm{d}k^{-1})^\dagger\big]^\top \circ k = \big[(\mathrm{d}k^{-1})^{\dagger\top}\nabla_{\mathbb{R}^2}(f \circ k^{-1})\big] \circ k.$$

We say $f \in \mathrm{H}^1(\partial\Omega)$, if additionally $f \in \mathrm{L}^2(\partial\Omega)$ and $\nabla_\tau f \in \mathrm{L}^2(\partial\Omega)^3$. We endow $\mathrm{H}^1(\partial\Omega)$ with the following norm

$$\|f\|_{\mathrm{H}^1(\partial\Omega)} = \sqrt{\|f\|^2_{\mathrm{L}^2(\partial\Omega)} + \|\nabla_\tau f\|^2_{\mathrm{L}^2(\partial\Omega)}}.$$

Note that for a.e. $\zeta \in \partial\Omega$ the tangential space is spanned by the columns of $\mathrm{d}k^{-1}(k(\zeta))$. We denote the space of all $\mathrm{L}^2(\partial\Omega)$ vector fields that are pointwise a.e. in the tangential space by

$$\mathrm{L}^2_\tau(\partial\Omega) := \{g \in \mathrm{L}^2(\partial\Omega)^3 \mid \nu \cdot g = 0\}.$$

By construction $\nabla_\tau f$ belongs to $\mathrm{L}^2_\tau(\partial\Omega)$. This can be seen by

$$(\nu \cdot \nabla_\tau f) \circ k^{-1} = \nu \circ k^{-1} \cdot \mathrm{d}k^{-1}\big((\mathrm{d}k^{-1})^\top \mathrm{d}k^{-1}\big)^{-1}\nabla_{\mathbb{R}^2}(f \circ k^{-1}) = 0,$$

because $\nu \circ k^{-1} \perp \mathrm{d}k^{-1}$ by definition.

The orthogonal projection on $\mathrm{L}^2_\tau(\partial\Omega)$ is given by $q \mapsto (\nu \times q) \times \nu$. For a $Q \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^3)^3$ we define the *tangential trace* by

$$\pi_\tau Q := \big(\nu \times Q\big|_{\partial\Omega}\big) \times \nu$$

For smooth functions $F \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^3)$ the next lemma shows that the tangential gradient on $\partial\Omega$ matches the tangential trace of the volume gradient on Ω.

Lemma 3.4. *For $F \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^3)$ we have $F\big|_{\partial\Omega} \in \mathrm{H}^1(\partial\Omega)$ and*

$$\nabla_\tau(F\big|_{\partial\Omega}) = \big(\nu \times (\nabla F)\big|_{\partial\Omega}\big) \times \nu = \pi_\tau \nabla F.$$

Proof. Note that $\operatorname{supp} F$ is compact by assumption, therefore also $\operatorname{supp} F\big|_{\partial\Omega}$ is compact and we need only finitely many charts to cover $\operatorname{supp} F\big|_{\partial\Omega}$. Consequently, it is enough to show that $F\big|_{\partial\Omega}$ is in $\mathrm{H}^1_{\mathrm{loc}}(\partial\Omega)$.

Let $k\colon \Gamma \to U$ be an arbitrary strongly Lipschitz chart. Then $F\big|_{\partial\Omega} \circ k^{-1} = F \circ k^{-1}$ belongs to $\mathrm{H}^1(U)$ by the chain rule. The tangential space at a.e. $\zeta \in \Gamma$ is given by the columns of $\mathrm{d}k^{-1}(k(\zeta))$. By construction the normal vector $\nu(\zeta)$ is orthogonal on this space. By Definition 3.3 and the chain rule we have

$$\big(\nabla_\tau F\big|_{\partial\Omega}\big)\big|_\Gamma = \big[\mathrm{d}(F \circ k^{-1})(\mathrm{d}k^{-1})^\dagger\big]^\top \circ k = \big[(\mathrm{d}F \circ k^{-1})\mathrm{d}k^{-1}(\mathrm{d}k^{-1})^\dagger\big]^\top \circ k.$$

Note that by Lemma C.3 the matrix $\mathrm{d}k^{-1}(\mathrm{d}k^{-1})^\dagger \circ k(\zeta)$ is the orthogonal projection on $\operatorname{ran}\mathrm{d}k^{-1}(k(\zeta))$ for a.e. $\zeta \in \Gamma$. In particular this matrix is symmetric. Moreover, by Lemma C.2 also $(\nu(\zeta) \times \cdot) \times \nu(\zeta)$ is the orthogonal projection on the same space for a.e. $\zeta \in \Gamma$. Hence,

$$\begin{aligned}\big(\nabla_\tau F\big|_{\partial\Omega}\big)\big|_\Gamma &= \big(\mathrm{d}k^{-1}(\mathrm{d}k^{-1})^\dagger \circ k\big)(\nabla F \circ k^{-1}) \circ k = \big(\mathrm{d}k^{-1}(\mathrm{d}k^{-1})^\dagger \circ k\big)(\nabla F)\big|_\Gamma \\ &= \big(\nu \times (\nabla F)\big|_\Gamma\big) \times \nu = (\pi_\tau \nabla F)\big|_\Gamma. \qquad \square\end{aligned}$$

Lemma 3.5. *Let $F \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^3)$ and $\Phi \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^3)^3$. Then*

$$\big\langle \pi_\tau \nabla F, \nu \times \Phi\big|_{\partial\Omega}\big\rangle_{\mathrm{L}^2(\partial\Omega)} = \big\langle F\big|_{\partial\Omega}, \nu \cdot (\operatorname{curl}\Phi)\big|_{\partial\Omega}\big\rangle_{\mathrm{L}^2(\partial\Omega)}.$$

Proof. By the integration by parts formula for curl and div-∇ we have

$$\begin{aligned}\left\langle \pi_\tau \nabla F, \nu \times \Phi\big|_{\partial\Omega}\right\rangle_{\mathrm{L}^2(\partial\Omega)} &= \langle \nabla F, \operatorname{curl}\Phi\rangle_{\mathrm{L}^2(\Omega)} - \langle \underbrace{\operatorname{curl}\nabla F}_{=0}, \Phi\rangle_{\mathrm{L}^2(\Omega)} \\ &= -\langle F, \underbrace{\operatorname{div}\operatorname{curl}\Phi}_{=0}\rangle_{\mathrm{L}^2(\Omega)} + \left\langle F\big|_{\partial\Omega}, \nu\cdot(\operatorname{curl}\Phi)\big|_{\partial\Omega}\right\rangle_{\mathrm{L}^2(\partial\Omega)} \\ &= \left\langle F\big|_{\partial\Omega}, \nu\cdot(\operatorname{curl}\Phi)\big|_{\partial\Omega}\right\rangle_{\mathrm{L}^2(\partial\Omega)}. \qquad ❑\end{aligned}$$

The previous lemma motivates the following alternative definition for $\mathrm{L}^2(\partial\Omega)$ elements that possess a tangential gradient in a weak sense.

Definition 3.6. Let Ω be a strongly Lipschitz domain. Then we say $f \in \tilde{\mathrm{H}}^1(\partial\Omega)$, if there exists a $q \in \mathrm{L}^2_\tau(\partial\Omega)$ such that for all $\Phi \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^3)^3$

$$\left\langle q, \nu\times\Phi\big|_{\partial\Omega}\right\rangle_{\mathrm{L}^2(\partial\Omega)} = \left\langle f, \nu\cdot(\operatorname{curl}\Phi)\big|_{\partial\Omega}\right\rangle_{\mathrm{L}^2(\partial\Omega)}.$$

Moreover, we say $\widetilde{\nabla}_\tau f = q$.

Our goal will be to show that the space $\tilde{\mathrm{H}}^1(\partial\Omega)$ coincides with $\mathrm{H}^1(\partial\Omega)$. By Lemma 3.5 we see that $F\big|_{\partial\Omega} \in \tilde{\mathrm{H}}^1(\partial\Omega)$ for every $F \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^3)$.

Theorem 3.7. *The set* $\left\{\Phi\big|_{\partial\Omega} \,\middle|\, \Phi \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^d)\right\}$ *is dense in* $\mathrm{H}^1(\partial\Omega)$ *w.r.t.* $\|\cdot\|_{\mathrm{H}^1(\partial\Omega)}$.

Proof. We will divide the proof into four steps. The first step is only needed, if $\partial\Omega$ is unbounded.

1. Step: *Reduce problem to finitely many charts.* Let $f \in \mathrm{H}^1(\partial\Omega)$. Then we can approximate f by a cutoff version of f w.r.t. $\|\cdot\|_{\mathrm{H}^1(\partial\Omega)}$ in the following way. For given $\epsilon > 0$ we choose $r > 0$ so large that for $\Gamma_r := \partial\Omega \cap \mathrm{B}_r(0)$ and $\Gamma_r^{\complement} = \partial\Omega \setminus \mathrm{B}_r(0)$

$$\|f\|_{\mathrm{L}^2(\Gamma_r^\complement)} + \|\nabla_\tau f\|_{\mathrm{L}^2(\Gamma_r^\complement)} = \left(\int_{\partial\Omega\setminus\mathrm{B}_r(0)} \|f\|^2 \,\mathrm{d}\mu\right)^{\frac12} + \left(\int_{\partial\Omega\setminus\mathrm{B}_r(0)} \|\nabla_\tau f\|^2 \,\mathrm{d}\mu\right)^{\frac12} < \frac{\epsilon}{6},$$

where $\mathrm{B}_r(0)$ is the ball in $\mathbb{R}^d$. We choose a cutoff $\chi \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^d)$ such that

$$0 \le \chi \le 1, \quad \|\nabla\chi\|_\infty \le 1, \quad \operatorname{supp}\chi \subseteq \mathrm{B}_{r+2}(0), \quad \text{and} \quad \chi\big|_{\mathrm{B}_r(0)} \equiv 1.$$

Then we define $f_r := \chi f$. It is easy to check that $f_r \in \mathrm{H}^1(\partial\Omega)$ and $\nabla_\tau f_r = (\nabla_\tau\chi)f + \chi\nabla_\tau f$. Hence, we have

$$\begin{aligned}\|f - f_r\|_{\mathrm{H}^1(\partial\Omega)} &\le \|f - f_r\|_{\mathrm{L}^2(\partial\Omega)} + \|\nabla_\tau f - \nabla_\tau f_r\|_{\mathrm{L}^2(\partial\Omega)} \\ &= \|f - \chi f\|_{\mathrm{L}^2(\partial\Omega)} + \|\nabla_\tau f - (\nabla_\tau\chi)f - \chi\nabla_\tau f\|_{\mathrm{L}^2(\partial\Omega)}\end{aligned}$$

Note that $\chi \equiv 1$ and $\nabla_\tau\chi \equiv 0$ on $\mathrm{B}_r(0)$, therefore, we further have

$$\begin{aligned}&= \|f - \chi f\|_{\mathrm{L}^2(\Gamma_r^\complement)} + \|\nabla_\tau f - (\nabla_\tau\chi)f - \chi\nabla_\tau f\|_{\mathrm{L}^2(\Gamma_r^\complement)} \\ &\le 3\big(\|f\|_{\mathrm{L}^2(\Gamma_r^\complement)} + \|\nabla_\tau f\|_{\mathrm{L}^2(\Gamma_r^\complement)}\big) \le \frac{\epsilon}{2}\end{aligned}$$

2. Step: *Approximate in local coordinates.* By the definition of a strongly Lipschitz domain we have for every $\zeta \in \partial\Omega$, a hyperplane W, a cylinder $C_{\epsilon,h}(\zeta)$ (ϵ and h depend on ζ), and a chart $k\colon \Gamma \to \mathrm{B}_\epsilon(0)$, where $\Gamma = \partial\Omega \cap C_{\epsilon,h}(\zeta)$. Hence, we can cover $\Gamma_r \subseteq \partial\Omega$ by $\bigcup_{\zeta\in\Gamma_r} C_{\epsilon,h}(\zeta)$ and since Ω is bounded, there is a finite subcover $\bigcup_{i=1}^m C_{\epsilon_i,h_i}(p_i)$. We employ a partition of unity and obtain $(\alpha_i)_{i=1}^m$, subordinate to this subcover, i.e.,

$$\alpha_i \in \mathring{\mathrm{C}}^\infty\big(C_{\epsilon_i,h_i}(p_i)\big), \quad \alpha_i(\zeta) \in [0,1], \quad \text{and} \quad \sum_{i=1}^m \alpha_i(\zeta) = 1 \quad \text{for all} \quad \zeta \in \partial\Omega.$$

For f_r we define $f_i = \alpha_i\big|_{\partial\Omega} f_r$. It is straightforward to show that also $f_i \in \mathrm{H}^1(\partial\Omega)$. We define $\Gamma_i = \partial\Omega \cap C_{\epsilon_i,h_i}(p_i)$ and the corresponding chart $k_i\colon \Gamma_i \to \mathrm{B}_{\epsilon_i}(0) \subseteq \mathbb{R}^{d-1}$.

Note that $\alpha_i\big|_{\partial\Omega}$ has compact support in Γ_i. Therefore, $f_i \circ k_i^{-1}$ has compact support in $\mathrm{B}_{\epsilon_i}(0)$ and $f_i \circ k_i^{-1} \in \mathring{\mathrm{H}}^1(\mathrm{B}_{\epsilon_i}(0))$. This implies that there exists a sequence $(\varphi_{i,n})_{n\in\mathbb{N}}$ in $\mathring{\mathrm{C}}^\infty(\mathrm{B}_{\epsilon_i}(0))$ that converges to $f_i \circ k_i^{-1}$ w.r.t. $\|\cdot\|_{\mathrm{H}^1(\mathrm{B}_{\epsilon_i}(0))}$.

3. Step: *Lift local approximation to* $\mathbb{R}^d$. We define an extension of $\varphi_{i,n}$ on $\mathbb{R}^d$ with support on a strip by

$$\Phi_{i,n}\left(\begin{bmatrix}\zeta_1\\ \vdots\\ \zeta_d\end{bmatrix}\right) \coloneqq \varphi_{i,n}\left(\begin{bmatrix}\zeta_1\\ \vdots\\ \zeta_{d-1}\end{bmatrix}\right) \quad \text{or} \quad \Phi_{i,n}(\zeta) \coloneqq \varphi_{i,n}(W^{\mathsf{T}}(\zeta - p_i)) \tag{1}$$

in the general coordinates. Hence, $\Phi_{i,n} \in \mathrm{C}^\infty(\mathbb{R}^d)$. Note that we do not want that $\operatorname{supp}\Phi_{i,n}$ intersects $\partial\Omega$ outside of Γ_i. Thus, we multiply $\Phi_{i,n}$ by a suitable $\mathring{\mathrm{C}}^\infty$ cutoff function that is 1 in a neighborhood of Γ_i (for all $n \in \mathbb{N}$ the same cutoff function). Consequently, we even have $\Phi_{i,n} \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^d)$.

By construction we have $\Phi_{i,n}\big|_{\Gamma_i} = \varphi_{i,n} \circ k_i$ and $\Phi_{i,n}\big|_{\partial\Omega} \to f_i$ in $\mathrm{H}^1(\partial\Omega)$. Now we define $\Phi_n = \sum_{i=1}^m \Phi_{i,n} \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^d)$ and obtain $\Phi_n\big|_{\partial\Omega} \to f_r$ in $\mathrm{H}^1(\partial\Omega)$.

4. Step: *Finish.* Finally, we choose $n \in \mathbb{N}$ so large that $\|f_r - \Phi_n\|_{\mathrm{H}^1(\partial\Omega)} \le \frac{\epsilon}{2}$. Then we have

$$\|f - \Phi_n\|_{\mathrm{H}^1(\partial\Omega)} \le \|f - f_r\|_{\mathrm{H}^1(\partial\Omega)} + \|f_r - \Phi_n\|_{\mathrm{H}^1(\partial\Omega)} \le \epsilon. \qquad \square$$

The density of $\{\Phi\big|_{\partial\Omega} \mid \Phi \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^d)\}$ implies that every $f \in \mathrm{H}^1(\partial\Omega)$ is automatically also in $\tilde{\mathrm{H}}^1(\partial\Omega)$, as the following corollary shows.

Corollary 3.8. $\mathrm{H}^1(\partial\Omega) \subseteq \tilde{\mathrm{H}}^1(\partial\Omega)$ *and* $\nabla_\tau f = \widetilde{\nabla}_\tau f$ *for all* $f \in \mathrm{H}^1(\partial\Omega)$.

Proof. Let $f \in \mathrm{H}^1(\partial\Omega)$. Then by Theorem 3.7 there exists a sequence $(F_n)_{n\in\mathbb{N}}$ in $\mathring{\mathrm{C}}^\infty(\mathbb{R}^3)$ such that $F_n\big|_{\partial\Omega} \to f$ w.r.t. $\|\cdot\|_{\mathrm{H}^1(\partial\Omega)}$. Hence, by Lemma 3.4 and Lemma 3.5 we have for every $\Phi \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^3)$

$$\begin{aligned}\langle \nabla_\tau f, \nu \times \Phi\rangle_{\mathrm{L}^2(\partial\Omega)} &= \lim_{n\to\infty} \langle \nabla_\tau F_n\big|_{\partial\Omega}, \nu \times \Phi\rangle_{\mathrm{L}^2(\partial\Omega)} = \lim_{n\to\infty} \langle \pi_\tau \nabla F_n, \nu \times \Phi\rangle_{\mathrm{L}^2(\partial\Omega)}\\ &= \lim_{n\to\infty} \langle F_n\big|_{\partial\Omega}, \nu \cdot (\operatorname{curl}\Phi)\big|_{\partial\Omega}\rangle_{\mathrm{L}^2(\partial\Omega)} = \langle f, \nu \cdot (\operatorname{curl}\Phi)\big|_{\partial\Omega}\rangle_{\mathrm{L}^2(\partial\Omega)},\end{aligned}$$

which implies $f \in \tilde{\mathrm{H}}^1(\partial\Omega)$ and $\nabla_\tau f = \widetilde{\nabla}_\tau f$. $\square$

The next two lemmas are the foundation of the main result (for general strongly Lipschitz charts their analogies are Lemmas A.1 and A.2). The second of these lemmas gives a lifting of a smooth function φ on a flat domain in $\mathbb{R}^2$ to a smooth function Φ on $\mathbb{R}^3$ such that the twisted tangential trace of the lifting Φ equals the tangential field that corresponds to φ (i.e., $\mathrm{d}k^{-1}\varphi$). This automatically gives an identity for the $\mathbb{R}^2$ divergence of φ in terms of Φ.

Lemma 3.9. *Let* $k\colon \Gamma \to U$ *be a strongly Lipschitz chart. Then for every* $\varphi \in \mathring{\mathrm{C}}^\infty(U)^2$ *we have*

$$\frac{1}{\sqrt{\det\left((\mathrm{d}k^{-1})^{\mathsf{T}}\mathrm{d}k^{-1}\right)}}\mathrm{d}k^{-1}\varphi = (\nu \circ k^{-1}) \times \begin{bmatrix}\varphi_2\\ -\varphi_1\\ 0\end{bmatrix}.$$

Proof. The following calculation proves the claim

$$\begin{aligned}
&(\nu \circ k^{-1}) \times \begin{bmatrix} \varphi_2 \\ -\varphi_1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -\nu_3 & \nu_2 \\ \nu_3 & 0 & -\nu_1 \\ -\nu_2 & \nu_1 & 0 \end{bmatrix} \circ k^{-1} \begin{bmatrix} \varphi_2 \\ -\varphi_1 \\ 0 \end{bmatrix} = \varphi_2 \begin{bmatrix} 0 \\ \nu_3 \\ -\nu_2 \end{bmatrix} \circ k^{-1} - \varphi_1 \begin{bmatrix} -\nu_3 \\ 0 \\ \nu_1 \end{bmatrix} \circ k^{-1} \\
&= \frac{1}{\sqrt{1 + \|\nabla a\|^2}} \left(\varphi_1 \begin{bmatrix} 1 \\ 0 \\ \partial_1 a \end{bmatrix} + \varphi_2 \begin{bmatrix} 0 \\ 1 \\ \partial_2 a \end{bmatrix} \right) = \frac{1}{\sqrt{\det\left((\mathrm{d}k^{-1})^\mathsf{T} \mathrm{d}k^{-1}\right)}} \mathrm{d}k^{-1} \varphi. \qquad \square
\end{aligned}$$

Lemma 3.10. *Let $\Gamma \subseteq \partial\Omega$ be a chart domain and $k\colon \Gamma \to U$ a strongly Lipschitz chart. Then for every $\varphi \in \mathring{\mathrm{C}}^\infty(U)^2$ there exists a $\Phi \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^3)^3$ such that we have*

$$\Phi\big|_\Gamma = \begin{bmatrix} \varphi_2 \\ -\varphi_1 \\ 0 \end{bmatrix} \circ k \quad \textit{and} \quad \Phi\big|_{\partial\Omega \setminus \Gamma} = 0$$

on the boundary, and

$$\mathrm{d}k^{-1} \varphi = \sqrt{\det\left((\mathrm{d}k^{-1})^\mathsf{T} \mathrm{d}k^{-1}\right)} (\nu \times \Phi) \circ k^{-1}, \tag{2}$$

$$\operatorname{div}_{\mathbb{R}^2} \varphi = -\sqrt{\det\left((\mathrm{d}k^{-1})^\mathsf{T} \mathrm{d}k^{-1}\right)} (\nu \cdot \operatorname{curl} \Phi) \circ k^{-1}. \tag{3}$$

Proof. We define

$$\hat{\Phi}\colon \begin{cases} U \times \mathbb{R} \subseteq \mathbb{R}^3 & \to \quad \mathbb{C}^3, \\ \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} & \mapsto \quad \begin{bmatrix} \varphi_2(\zeta_1, \zeta_2) \\ -\varphi_1(\zeta_1, \zeta_2) \\ 0 \end{bmatrix}. \end{cases}$$

Since φ has compact support in U we can extend $\hat{\Phi}$ outside of $U \times \mathbb{R}$ by 0. Moreover we choose an $\epsilon > 0$ such that the ball with radius 2ϵ around Γ satisfies

$$\mathrm{B}_{2\epsilon}(\Gamma) \cap \operatorname{supp} \hat{\Phi} \cap (\partial\Omega \setminus \Gamma) = \emptyset.$$

Finally, we choose a cutoff function $\chi \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^3)$ such that $\chi\big|_{\mathrm{B}_\epsilon(\Gamma)} = 1$ and $\chi\big|_{\mathrm{B}_{2\epsilon}(\Gamma)^\complement} = 0$ and we define $\Phi \coloneqq \chi\hat{\Phi}$. Hence, $\Phi\big|_{\partial\Omega\setminus\Gamma} = 0$.

By construction we have $\Phi \circ k^{-1}(x_1, x_2) = \begin{bmatrix} \varphi_2(x_1,x_2) \\ -\varphi_1(x_1,x_2) \\ 0 \end{bmatrix}$. Thus, Lemma 3.9 implies (2).

Note that locally around Γ we have $\Phi = \begin{bmatrix} \varphi_2 \\ -\varphi_1 \\ 0 \end{bmatrix}$ and $\partial_3 \Phi = 0$. Hence, we have

$$\begin{aligned}
-\sqrt{1 + \|\nabla a\|^2}\, \nu \cdot \operatorname{curl} \Phi &= \begin{bmatrix} \partial_1 a \\ \partial_2 a \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix} \begin{bmatrix} \varphi_2 \\ -\varphi_1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \partial_1 a \\ \partial_2 a \\ -1 \end{bmatrix} \cdot \begin{bmatrix} \partial_3 \varphi_1 \\ \partial_3 \varphi_2 \\ -\partial_1 \varphi_1 - \partial_2 \varphi_2 \end{bmatrix} \\
&= \partial_1 a \underbrace{\partial_3 \varphi_1}_{=0} + \partial_2 a \underbrace{\partial_3 \varphi_2}_{=0} + \partial_1 \varphi_1 + \partial_2 \varphi_2 = \operatorname{div}_{\mathbb{R}^2} \varphi. \qquad \square
\end{aligned}$$

Finally, we come to the main result, that proves that both presented approaches (Definition 3.3 and Definition 3.6) to the first order Sobelev space on $\partial\Omega$ lead to the same space.

Theorem 3.11. *$\tilde{\mathrm{H}}^1(\partial\Omega) = \mathrm{H}^1(\partial\Omega)$ and $\widetilde{\nabla}_\tau f = \nabla_\tau f$ for all $f \in \mathrm{H}^1(\partial\Omega)$.*

Proof. We have already shown $\mathrm{H}^1(\partial\Omega) \subseteq \tilde{\mathrm{H}}^1(\partial\Omega)$ in Corollary 3.8. Hence, it is left to show the reverse inclusion.

Let $f \in \tilde{\mathrm{H}}^1(\partial\Omega)$, i.e., there exists a $q \in \mathrm{L}^2_\tau(\partial\Omega)^3$ such that

$$\langle q, \nu \times \Phi\big|_{\partial\Omega}\rangle_{\mathrm{L}^2(\partial\Omega)} = \langle f, \nu \cdot (\operatorname{curl}\Phi)\big|_{\partial\Omega}\rangle_{\mathrm{L}^2(\partial\Omega)} \quad \text{for all} \quad \Phi \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^3)^3 \tag{4}$$

Let $\Gamma \subseteq \partial\Omega$ be a chart domain, $U \subseteq \mathbb{R}^2$ open and $k\colon \Gamma \to U$ a strongly Lipschitz chart. For an arbitrary $\varphi \in \mathring{\mathrm{C}}^\infty(U)$ we define Φ as in Lemma 3.10. Then we have

$$\begin{aligned}
&-\langle f \circ k^{-1}, \operatorname{div}_{\mathbb{R}^2}\varphi\rangle_{\mathrm{L}^2(U)} \\
&\quad\overset{(3)}{=} \left\langle f \circ k^{-1}, \sqrt{\det\left((\mathrm{d}k^{-1})^\top \mathrm{d}k^{-1}\right)}(\nu \cdot \operatorname{curl}\Phi) \circ k^{-1}\right\rangle_{\mathrm{L}^2(U)} \\
&\quad= \big\langle f, \nu \cdot \operatorname{curl}\Phi\big|_{\partial\Omega}\big\rangle_{\mathrm{L}^2(\Gamma)} = \big\langle f, \nu \cdot \operatorname{curl}\Phi\big|_{\partial\Omega}\big\rangle_{\mathrm{L}^2(\partial\Omega)} \overset{(4)}{=} \big\langle q, \nu \times \Phi\big|_{\partial\Omega}\big\rangle_{\mathrm{L}^2(\partial\Omega)} \\
&\quad= \big\langle q, \nu \times \Phi\big|_{\partial\Omega}\big\rangle_{\mathrm{L}^2(\Gamma)} = \left\langle q \circ k^{-1}, \sqrt{\det\left((\mathrm{d}k^{-1})^\top \mathrm{d}k^{-1}\right)}(\nu \times \Phi) \circ k^{-1}\right\rangle_{\mathrm{L}^2(U)} \\
&\quad\overset{(2)}{=} \big\langle q \circ k^{-1}, \mathrm{d}k^{-1}\varphi\big\rangle_{\mathrm{L}^2(U)} = \big\langle (\mathrm{d}k^{-1})^\top (q \circ k^{-1}), \varphi\big\rangle_{\mathrm{L}^2(U)}
\end{aligned}$$

Hence, $f \circ k^{-1} \in \mathrm{H}^1(U)$ and $\nabla_\tau f\big|_\Gamma = q\big|_\Gamma$. Since this is true for any chart k we conclude $f \in \mathrm{H}^1_{\mathrm{loc}}(\partial\Omega)$ and $\nabla_\tau f = q = \widetilde{\nabla}_\tau f$. Since $q \in \mathrm{L}^2_\tau(\partial\Omega)$ we conclude $f \in \mathrm{H}^1(\partial\Omega)$. ❑

4. Back to the original question

In order to verify Claim A we will recall the basics about $\mathrm{H}(\operatorname{curl}, \Omega)$, see, e.g., [Mon03, Section 3.5].

First of all, we define the space

$$\mathrm{H}(\operatorname{curl}, \Omega) \coloneqq \{E \in \mathrm{L}^2(\Omega)^3 \mid \operatorname{curl} E \in \mathrm{L}^2(\Omega)^3\}.$$

where we understand $\operatorname{curl} E$ a priori in a distributional sense. Note that a straightforward calculation gives $\operatorname{curl}\nabla F = 0$ for all $F \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^3)$. Hence, by continuity this passes on to $F \in \mathrm{H}^1(\Omega)$. This leads to $\nabla F \in \mathrm{H}(\operatorname{curl}, \Omega)$ for all $F \in \mathrm{H}^1(\Omega)$. The integration by parts formula for curl for smooth functions $E, H \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^3)^3$ reads as follows

$$\langle E, \operatorname{curl} H\rangle_{\mathrm{L}^2(\Omega)} - \langle \operatorname{curl} E, H\rangle_{\mathrm{L}^2(\Omega)} = \big\langle \pi_\tau E, \nu \times H\big|_{\partial\Omega}\big\rangle_{\mathrm{L}^2(\partial\Omega)}.$$

This motivates the following weak definition of L^2 tangential traces for $\mathrm{H}(\operatorname{curl}, \Omega)$ elements.

Definition 4.1. We say $E \in \mathrm{H}(\operatorname{curl}, \Omega)$ possesses a (weak) L^2 tangential trace, if there exists a $q \in \mathrm{L}^2_\tau(\partial\Omega)$ such that

$$\langle E, \operatorname{curl}\Phi\rangle_{\mathrm{L}^2(\Omega)} - \langle \operatorname{curl} E, \Phi\rangle_{\mathrm{L}^2(\Omega)} = \langle q, \nu \times \Phi\big|_{\partial\Omega}\rangle_{\mathrm{L}^2_\tau(\partial\Omega)} \quad \text{for all} \quad \Phi \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^3)^3.$$

We say then q is the (weak) tangential trace of E, i.e., $\pi_\tau E = q$.

Theorem 4.2. *Let $F \in \mathrm{H}^1(\Omega)$ be such that ∇F possesses a (weak) L^2 tangential trace. Then $F\big|_{\partial\Omega} \in \mathrm{H}^1(\partial\Omega)$ and $\pi_\tau \nabla F = \nabla_\tau F\big|_{\partial\Omega}$.*

Proof. Let $q \in \mathrm{L}^2_\tau(\partial\Omega)$ be such that $q = \pi_\tau \nabla F$. By the integration by parts formula for curl and div-∇, we have for an arbitrary $\Phi \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^3)^3$

$$\begin{aligned}
\langle q, \nu \times \Phi\big|_{\partial\Omega}\rangle_{\mathrm{L}^2_\tau(\partial\Omega)} &= \langle \nabla F, \operatorname{curl}\Phi\rangle_{\mathrm{L}^2(\Omega)} - \langle \underbrace{\operatorname{curl}\nabla F}_{=0}, \Phi\rangle_{\mathrm{L}^2(\Omega)} \\
&= -\langle F, \underbrace{\operatorname{div}\operatorname{curl}\Phi}_{=0}\rangle_{\mathrm{L}^2(\Omega)} + \left\langle F\big|_{\partial\Omega}, \nu\cdot(\operatorname{curl}\Phi)\big|_{\partial\Omega}\right\rangle_{\mathrm{L}^2(\partial\Omega)} \\
&= \left\langle F\big|_{\partial\Omega}, \nu\cdot(\operatorname{curl}\Phi)\big|_{\partial\Omega}\right\rangle_{\mathrm{L}^2(\partial\Omega)}.
\end{aligned}$$

Hence, $F\big|_{\partial\Omega}$ satisfies all requirements of Definition 3.6, which implies, by Theorem 3.11, $F\big|_{\partial\Omega} \in \mathrm{H}^1(\partial\Omega)$. In particular we have

$$\pi_\tau \nabla F = q = \widetilde{\nabla}_\tau F\big|_{\partial\Omega} = \nabla_\tau F\big|_{\partial\Omega}. \qquad \square$$

5. Conclusion

With Theorem 3.11 we have shown that both presented approaches to $\mathrm{H}^1(\partial\Omega)$ agree. Moreover, Theorem 4.2 answers the question about the validity of Claim A, that started the whole discussion, positively. Hence, we provide the details that are used in [Cos90, Proof of Thm. 2], [BBBCD97, Section Le cas tridimensionnel], [Mon03, Proof of Lem. 3.53], and [BCS02, Proof of Thm. 5.1].

Acknowledgement

We thank Dirk Pauly and Martin Costabel for the discussions about Claim A.

Appendix A. Details for general hyperplanes

Note that in the setting with a general hyperplane $W = \operatorname{span}\{w_1, w_2\}$, where w_1 and w_2 are an orthonormal basis of W, and its normal vector v we have

$$k^{-1}\colon \begin{cases} U \subseteq \mathbb{R}^2 & \to \quad \Gamma, \\ (x_1, x_2) & \mapsto \quad p + x_1 w_1 + x_2 w_2 + a(x_1, x_2)v. \end{cases}$$

Hence,

$$\mathrm{d}k^{-1} = \begin{bmatrix} w_1 + \partial_1 a v & w_2 + \partial_2 a v \end{bmatrix}$$

and the normal vector on the tangential space is locally given by

$$\nu \circ k^{-1} = \frac{1}{\sqrt{1 + \|\nabla a\|^2}}(-\partial_1 a w_1 - \partial_2 a w_2 + v).$$

Moreover, we have

$$(\mathrm{d}k^{-1})^\mathsf{T} \mathrm{d}k^{-1} = \begin{bmatrix} 1 + (\partial_1 a)^2 & \partial_1 a \partial_2 a \\ \partial_1 a \partial_2 a & 1 + (\partial_2 a)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \partial_1 a \\ \partial_2 a \end{bmatrix} \begin{bmatrix} \partial_1 a & \partial_2 a \end{bmatrix}.$$

and therefore Lemma 3.1 follows also for general strongly Lipschitz charts:

$$\det\left((\mathrm{d}k^{-1})^\mathsf{T} \mathrm{d}k^{-1}\right) = 1 + \|\nabla a\|^2.$$

We show the modified Lemmas 3.9 and 3.10 for general strongly Lipschitz charts

Lemma A.1. *For $\varphi \in \mathring{\mathrm{C}}^\infty(U)^2$ we have*

$$\mathrm{d}k^{-1}\varphi = \sqrt{\det\left((\mathrm{d}k^{-1})^\mathsf{T} \mathrm{d}k^{-1}\right)}\,(\nu \circ k^{-1}) \times (\varphi_2 w_1 - \varphi_1 w_2),$$

where the orthogonal basis $\{w_1, w_2, v\}$ is chosen such that $w_1 \times w_2 = v$ (if this is not already true we relabel w_1 and w_2).

Note that $w_1 \times w_2 = v$ implies

$$w_1 \times v = -w_2 \quad \text{and} \quad w_2 \times v = w_1.$$

Proof. Note that

$$\sqrt{\det\big((\mathrm{d}k^{-1})^\mathsf{T}\mathrm{d}k^{-1}\big)}(\nu\circ k^{-1}) = -\partial_1 a w_1 - \partial_2 a w_2 + v.$$

Therefore, the following proves the claim:

$$\begin{aligned}(-\partial_1 a w_1 - \partial_2 a w_2 + v)\times(\varphi_2 w_1 - \varphi_1 w_2)&\\ &= (\partial_2 a v + w_2)\varphi_2 + (\partial_1 a v + w_1)\varphi_1 = \mathrm{d}k^{-1}\varphi. \quad ❑\end{aligned}$$

Lemma A.2. *Let $\Gamma\subseteq\partial\Omega$ be a chart domain and $k\colon\Gamma\to U$ a strongly Lipschitz chart. Then for every $\varphi\in\mathring{\mathrm{C}}^\infty(U)^2$ there exists a $\Phi\in\mathring{\mathrm{C}}^\infty(\mathbb{R}^3)^3$ such that we have*

$$\Phi\big|_\Gamma = W\begin{bmatrix}\varphi_2\\-\varphi_1\end{bmatrix}\circ k = (\varphi_2\circ k)w_1 - (\varphi_1\circ k)w_2 \quad\text{and}\quad \Phi\big|_{\partial\Omega\setminus\Gamma} = 0$$

on the boundary, and

$$\mathrm{d}k^{-1}\varphi = \sqrt{\det\big((\mathrm{d}k^{-1})^\mathsf{T}\mathrm{d}k^{-1}\big)}(\nu\times\Phi)\circ k^{-1}, \tag{5}$$

$$\operatorname{div}_{\mathbb{R}^2}\varphi = -\sqrt{\det\big((\mathrm{d}k^{-1})^\mathsf{T}\mathrm{d}k^{-1}\big)}(\nu\cdot\operatorname{curl}\Phi)\circ k^{-1}. \tag{6}$$

Proof. We define $\hat\Phi\in\mathrm{C}^\infty(\mathbb{R}^3)^3$ by

$$\hat\Phi(\zeta) = W\begin{bmatrix}\varphi_2\\-\varphi_1\end{bmatrix}\big(W^\mathsf{T}(\zeta-p)\big) = \varphi_2(W^\mathsf{T}(\zeta-p))w_1 - \varphi_1(W^\mathsf{T}(\zeta-p))w_2,$$

where $W\in\mathbb{R}^{3\times 2}$ is the matrix containing the vectors w_1 and w_2 as rows, i.e., $W=\begin{bmatrix}w_1 & w_2\end{bmatrix}$. Finally, we define $\Phi\in\mathring{\mathrm{C}}^\infty(\mathbb{R}^3)^3$ by $\chi\hat\Phi$ where $\chi\in\mathring{\mathrm{C}}^\infty(\mathbb{R}^3)$ is such that in a small neighborhood of Γ $\chi=1$ and $\Phi\big|_{\partial\Omega\setminus\Gamma}=0$. Basically, by construction we have $\Phi\big|_\Gamma = W\big[\begin{smallmatrix}\varphi_2\\-\varphi_1\end{smallmatrix}\big]\circ k$. Hence, we have $\Phi\circ k^{-1}=\varphi_2 w_1-\varphi_1 w_2$ and Lemma A.1 gives (5)

For an arbitrary $f\in\mathring{\mathrm{C}}^\infty(U)$ we have

$$\begin{aligned}-\langle f,\operatorname{div}_{\mathbb{R}^2}\varphi\rangle_{\mathrm{L}^2(U)} &= \langle\nabla_{\mathbb{R}^2}f,\varphi\rangle_{\mathrm{L}^2(U)} = \Big\langle\big((\mathrm{d}k^{-1})^\dagger\big)^\mathsf{T}\nabla_{\mathbb{R}^2}f,\mathrm{d}k^{-1}\varphi\Big\rangle_{\mathrm{L}^2(U)}\\ &= \Big\langle\big[(\mathrm{d}k^{-1})^{\dagger\mathsf{T}}\nabla_{\mathbb{R}^2}f\big]\circ k, \Big[\tfrac{1}{\sqrt{\det((\mathrm{d}k^{-1})^\mathsf{T}\mathrm{d}k^{-1})}}\mathrm{d}k^{-1}\varphi\Big]\circ k\Big\rangle_{\mathrm{L}^2(\Gamma)}\\ &= \big\langle\nabla_\tau(f\circ k),\nu\times\Phi\big|_\Gamma\big\rangle_{\mathrm{L}^2(\Gamma)} = \big\langle f\circ k,\nu\cdot(\operatorname{curl}\Phi)\big|_\Gamma\big\rangle_{\mathrm{L}^2(\Gamma)}\\ &= \Big\langle f,(\nu\cdot\operatorname{curl}\Phi)\circ k^{-1}\sqrt{\det\big((\mathrm{d}k^{-1})^\mathsf{T}\mathrm{d}k^{-1}\big)}\Big\rangle_{\mathrm{L}^2(U)}\end{aligned}$$

By density of $\mathring{\mathrm{C}}^\infty(U)$ in $\mathrm{L}^2(U)$ we obtain (6). ❑

Appendix B. Independence of the charts

Note that for two strongly Lipschitz charts $k_1\colon\Gamma_1\to U_1$, $k_2\colon\Gamma_2\to U_2$ with overlapping chart domains (i.e., $\Gamma_1\cap\Gamma_2\neq\emptyset$) we have that the columns of $\mathrm{d}k_1^{-1}(k_1(\zeta))$ and the columns of $\mathrm{d}k_2^{-1}(k_2(\zeta))$ span the same linear subspace of $\mathbb{R}^d$ for a.e. $\zeta\in\Gamma_1\cap\Gamma_2$, namely the *tangential space* of $\partial\Omega$ at ζ. The next lemma will specify this.

Lemma B.1. *Let $k_1\colon\Gamma_1\to U_1$ and $k_2\colon\Gamma_2\to U_2$ be strongly Lipschitz charts. Then*

$$\operatorname{ran}\big[\mathrm{d}k_1^{-1}\big(k_1(\zeta)\big)\big] = \operatorname{ran}\big[\mathrm{d}k_2^{-1}\big(k_2(\zeta)\big)\big] \quad\textit{for a.e.}\quad \zeta\in\Gamma_1\cap\Gamma_2.$$

Moreover,

$$(\mathrm{d}k_1^{-1})^\dagger\circ(k_1\circ k_2^{-1})\,\mathrm{d}k_2^{-1} = \mathrm{d}(k_1\circ k_2^{-1}). \tag{7}$$

Proof. The first assertion follows from

$$\mathrm{d}k_2^{-1} = \mathrm{d}(k_1^{-1} \circ k_1 \circ k_2^{-1}) = (\mathrm{d}k_1^{-1}) \circ (k_1 \circ k_2^{-1})\, \mathrm{d}(k_1 \circ k_2^{-1}) \tag{8}$$

and the fact that $\mathrm{d}(k_1 \circ k_2^{-1})(\zeta)$ is a regular matrix for a.e. $\zeta \in \Gamma_1 \cap \Gamma_2$. Multiplying both side of (8) from left with $(\mathrm{d}k_1^{-1})^\dagger \circ (k_1 \circ k_2^{-1})$ implies (7). ❑

Lemma B.2. *Let $k_1 \colon \Gamma_1 \to U_1$, $k_2 \colon \Gamma_2 \to U_2$ strongly Lipschitz charts. Then for a.e. $\zeta \in \Gamma_1 \cap \Gamma_2$ the following holds*

$$(\mathrm{d}k_1^{-1})(\mathrm{d}k_1^{-1})^\dagger \circ k_1(\zeta) = (\mathrm{d}k_2^{-1})(\mathrm{d}k_2^{-1})^\dagger \circ k_2(\zeta).$$

Proof. Note that for a.e. $\zeta \in \Gamma_1 \cap \Gamma_2$ we have $\operatorname{ran}[\mathrm{d}k_1^{-1}(k_1(\zeta))] = \operatorname{ran}[\mathrm{d}k_2^{-1}(k_2(\zeta))]$. By Lemma C.3 $(\mathrm{d}k_1^{-1})(\mathrm{d}k_1^{-1})^\dagger \circ k_1(\zeta)$ is the orthogonal projection on $\operatorname{ran}[\mathrm{d}k_1^{-1}(k_1(\zeta))]$ and $(\mathrm{d}k_2^{-1})(\mathrm{d}k_2^{-1})^\dagger \circ k_2(\zeta)$ is the orthogonal projection on $\operatorname{ran}[\mathrm{d}k_2^{-1}(k_2(\zeta))]$. Since these ranges coincide we conclude the assertion. ❑

Sometimes it is more convenient to work with the boundary derivative d_τ instead of the the tangential gradient ∇_τ. This derivative is given by $\mathrm{d}_\tau f = (\nabla_\tau f)^\mathsf{T}$ or locally by $(\mathrm{d}_\tau f)\big|_\Gamma = \big[\mathrm{d}(f \circ k^{-1})(\mathrm{d}k^{-1})^\dagger\big] \circ k$.

Proposition B.3. *Let $f \in \mathrm{H}^1(\partial\Omega)$. Then $\nabla_\tau f$ and $\mathrm{d}_\tau f$ are independent of the charts.*

Proof. Let k_1 and k_2 be two charts with overlapping chart domains. Then we have

$$\begin{aligned}
(\mathrm{d}_\tau f)\big|_{\Gamma_1 \cap \Gamma_2} &= \big[\mathrm{d}(f \circ k_2^{-1})(\mathrm{d}k_2^{-1})^\dagger\big] \circ k_2 = \big[\mathrm{d}(f \circ k_1^{-1} \circ k_1 \circ k_2^{-1})(\mathrm{d}k_2^{-1})^\dagger\big] \circ k_2 \\
&= \big[\mathrm{d}(f \circ k_1^{-1}) \circ (k_1 \circ k_2^{-1}) \underbrace{\mathrm{d}(k_1 \circ k_2^{-1})}_{\overset{(7)}{=}(\mathrm{d}k_1^{-1})^\dagger \circ (k_1 \circ k_2^{-1})\mathrm{d}k_2^{-1}}(\mathrm{d}k_2^{-1})^\dagger\big] \circ k_2 \\
&= \big[\mathrm{d}(f \circ k_1^{-1}) \circ (k_1 \circ k_2^{-1})(\mathrm{d}k_1^{-1})^\dagger \circ (k_1 \circ k_2^{-1}) \underbrace{\mathrm{d}k_2^{-1}(\mathrm{d}k_2^{-1})^\dagger}_{[\mathrm{d}k_1^{-1}(\mathrm{d}k_1^{-1})^\dagger] \circ (k_1 \circ k_2^{-1}) \overset{\text{L.B.2}}{=}}\big] \circ k_2
\end{aligned}$$

Note that $A^\dagger A A^\dagger = A^\dagger$.

$$\begin{aligned}
&= \big[\mathrm{d}(f \circ k_1) \circ (k_1 \circ k_2^{-1})(\mathrm{d}k_1^{-1})^\dagger \circ (k_1 \circ k_2^{-1})\big] \circ k_2 \\
&= \big[\mathrm{d}(f \circ k_1)(\mathrm{d}k_1^{-1})^\dagger\big] \circ k_1.
\end{aligned}$$

❑

Proposition B.4. *The surface measure on $\partial\Omega$ is independent of the partition and the charts.*

Proof. It is enough to show that two charts $k_1 \colon \Gamma_1 \to U_1$ and $k_2 \colon \Gamma_2 \to U_2$ with intersecting chart domains define the same surface measure on the intersection $\Gamma_1 \cap \Gamma_2$. The rest can be done by intersecting the two partitions.

We define the mapping

$$T \colon \begin{cases} k_2(\Gamma_1 \cap \Gamma_2) \subseteq U_2 & \to \quad k_1(\Gamma_1 \cap \Gamma_2) \subseteq U_1, \\ \qquad\qquad\quad x & \mapsto \quad (k_1 \circ k_2^{-1})(x), \end{cases}$$

which gives a bijective bi-Lipschitz continuous mapping. Note that by the chain rule we have

$$\mathrm{d}k_2^{-1} = \mathrm{d}(k_1^{-1} \circ k_1 \circ k_2^{-1}) = (\mathrm{d}k_1^{-1}) \circ (k_1 \circ k_2^{-1})\mathrm{d}(k_1 \circ k_2^{-1}) = (\mathrm{d}k_1^{-1}) \circ T \mathrm{d}T.$$

Moreover, by properties of the determinant we have

$$\begin{aligned}
|\det \mathrm{d}T|\sqrt{\det(\mathrm{d}k_1^{-1}\circ T)^\mathsf{T}(\mathrm{d}k_1^{-1}\circ T)} &= \sqrt{\det(\mathrm{d}T)^\mathsf{T}\mathrm{d}T}\sqrt{\det(\mathrm{d}k_1^{-1}\circ T)^\mathsf{T}(\mathrm{d}k_1^{-1}\circ T)}\\
&= \sqrt{\det(\mathrm{d}T)^\mathsf{T}(\mathrm{d}k_1^{-1}\circ T)^\mathsf{T}(\mathrm{d}k_1^{-1}\circ T)\mathrm{d}T}\\
&= \sqrt{\det((\mathrm{d}k_1^{-1}\circ T)\mathrm{d}T)^\mathsf{T}((\mathrm{d}k_1^{-1}\circ T)\mathrm{d}T)}\\
&= \sqrt{\det(\mathrm{d}k_2^{-1})^\mathsf{T}\mathrm{d}k_2^{-1}}.
\end{aligned}$$

Now for $\Upsilon \subseteq \Gamma_1 \cap \Gamma_2$ we have by change of variables

$$\begin{aligned}
\int_{k_1(\Upsilon)} \sqrt{\det(\mathrm{d}k_1^{-1})^\mathsf{T}\mathrm{d}k_1^{-1}}\,\mathrm{d}\lambda_{d-1} &= \int_{T^{-1}(k_1(\Upsilon))} \sqrt{\det(\mathrm{d}k_1^{-1})^\mathsf{T}\mathrm{d}k_1^{-1}}\circ T|\det \mathrm{d}T|\,\mathrm{d}\lambda_{d-1}\\
&= \int_{k_2(\Upsilon)} \sqrt{\det(\mathrm{d}k_2^{-1})^\mathsf{T}\mathrm{d}k_2^{-1}}\,\mathrm{d}\lambda_{d-1}.
\end{aligned}$$

Hence, the surface measure $\mu(\Upsilon)$ is independent of the charts. ❑

Appendix C. Some auxiliary lemmas

Lemma C.1. *Let $v \in \mathbb{R}^d$ then*

$$\det(I + vv^\mathsf{T}) = 1 + \|v\|^2.$$

Proof. Note that the determinant of a matrix equals the product of all eigenvalues. Let $b_1, \ldots, b_{d-1}$ denote an orthonormal basis of $\{v\}^\perp$. Then we can easily see that each b_i is an eigenvector of $I + vv^\mathsf{T}$ with eigenvalue 1. Furthermore, $(I + vv^\mathsf{T})v = (1 + \|v\|^2)v$ implies that v is an eigenvector with eigenvalue $1 + \|v\|^2$. Hence, we have found all eigenvalues and consequently the determinant equals $1 + \|v\|^2$. ❑

Lemma C.2. *For $w \in \mathbb{C}^3$ with $\|w\| = 1$ the mapping $A\colon v \mapsto (w \times v) \times w$ is the orthogonal projection on the orthogonal complement of* $\operatorname{span}\{w\}$.

Proof. Note that $(w \times v) \times w = -w \times (w \times v)$ and $w \times v = \left[\begin{smallmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{smallmatrix}\right] v$. Therefore,

$$(w \times v) \times w = -\begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}^2 v = \begin{bmatrix} w_2^2 + w_3^2 & -w_1w_2 & -w_1w_3 \\ -w_1w_2 & w_1^2 + w_3^2 & -w_2w_3 \\ -w_1w_3 & -w_2w_3 & w_1^2 + w_2^2 \end{bmatrix} v$$

Since $\|w\| = 1$ we further have

$$= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} w_1^2 & w_1w_2 & w_1w_3 \\ w_1w_2 & w_2^2 & w_2w_3 \\ w_1w_3 & w_2w_3 & w_3^2 \end{bmatrix}\right) v = (I - ww^\mathsf{T})v,$$

which shows the claim. ❑

Lemma C.3. *Let A be an injective matrix and $A^\dagger = (A^\mathsf{T}A)^{-1}A^\mathsf{T}$ its Moore-Penrose inverse. Then $AA^\dagger$ is the orthogonal projection on* $\operatorname{ran} A$.

Proof. Note that $\ker A^\mathsf{T} = (\operatorname{ran} A)^\perp$, $\ker A = (\operatorname{ran} A^\mathsf{T})^\perp$, and $\ker A^\dagger = \ker A^\mathsf{T}$. Therefore, $\ker AA^\dagger = \ker A^\mathsf{T} = (\operatorname{ran} A)^\perp$. Moreover,

$$AA^\dagger A = A(A^\mathsf{T}A)^{-1}A^\mathsf{T}A = A,$$

which implies that the $\operatorname{ran} A$ is invariant under $AA^\dagger$. Consequently $AA^\dagger$ is an orthogonal projection on $\operatorname{ran} A$. ❑

References

[AFP00] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000. doi:10.1093/oso/9780198502456.001.0001.

[BBBCD97] Faker Ben Belgacem, Christine Bernardi, Martin Costabel, and Monique Dauge. A density result for Maxwell's equations. *C. R. Acad. Sci., Paris, Sér. I*, 324(6):731–736, 1997. doi:10.1016/S0764-4442(97)86998-4.

[BCS02] A. Buffa, M. Costabel, and D. Sheen. On traces for $\mathbf{H}(\operatorname{curl}, \Omega)$ in Lipschitz domains. *J. Math. Anal. Appl.*, 276(2):845–867, 2002. doi:10.1016/S0022-247X(02)00455-9.

[BPS16] Sebastian Bauer, Dirk Pauly, and Michael Schomburg. The Maxwell compactness property in bounded weak Lipschitz domains with mixed boundary conditions. *SIAM J. Math. Anal.*, 48(4):2912–2943, 2016. doi:10.1137/16M1065951.

[Cos90] Martin Costabel. A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains. *Math. Methods Appl. Sci.*, 12(4):365–368, 1990. doi:10.1002/mma.1670120406.

[Gri85] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[Mon03] Peter Monk. *Finite element methods for Maxwell's equations*. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2003. doi:10.1093/acprof:oso/9780198508885.001.0001.

[Nag85] Jun-iti Nagata. *Modern general topology*, volume 33 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1985.

[Neč12] Jindřich Nečas. *Direct methods in the theory of elliptic equations*. Springer Monographs in Mathematics. Springer, Heidelberg, 2012. doi:10.1007/978-3-642-10455-8.

TU Bergakademie Freiberg, Institute of Applied Analysis, Akademiestrasse 6, D-09596 Freiberg, Germany

Email address: nathanael.skrepek@math.tu-freiberg.de