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Integral Equations and Operator Theory

Quasi Gelfand triples

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Abstract. We generalize the notion of Gelfand triples (also called Banach-Gelfand triples or rigged Hilbert spaces) by dropping the necessity of a continuous embedding. This means in our setting we lack of a chain inclusion. We replace the continuous embedding by a closed embedding of a dense subspace. This notion will be called *quasi Gelfand triple*. These triples appear naturally, when we regard the boundary spaces of spatially multidimensional differential operators, e.g., the Maxwell operator. We will show that there is a smallest space where we can continuously embed the entire triple. Moreover, we will show density results for intersections of members of the quasi Gelfand triple. Finally, we show that every quasi Gelfand triple can be decomposed into two "ordinary" Gelfand triples.

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1. Introduction

Normally, when we talk about Gelfand triples we have a Hilbert space $\mathcal{X}_0$ and a reflexive Banach space $\mathcal{X}_+$ that can be continuously and densely embedded into $\mathcal{X}_0$. The third space $\mathcal{X}_-$ is given by the completion of $\mathcal{X}_0$ with respect to

$$\|g\|_{\mathcal{X}_-} := \sup_{f\in\mathcal{X}_+\setminus\{0\}} \frac{|\langle g, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}}.$$

The duality between $\mathcal{X}_+$ and $\mathcal{X}_-$ is given by

$$\langle g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} = \lim_{k\to\infty} \langle g_k, f\rangle_{\mathcal{X}_0},$$

where $(g_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{X}_0$ that converges to g in $\mathcal{X}_-$. The space $\mathcal{X}_-$ is then isometrically isomorphic to $\mathcal{X}_+'$. The theory of Gelfand triples was introduced by I.M. Gelfand and A.G. Kostyuchenko [8]. The concept has been refined over time. In the introduction of [5] they give a short historical overview of Gelfand triples.

We want to weaken the assumptions such that the norm of $\mathcal{X}_+$ is not necessarily related to the norm of $\mathcal{X}_0$. Hence, we cannot expect a continuous embedding of $\mathcal{X}_+$ into $\mathcal{X}_0$. However, we still want to construct the dual $\mathcal{X}_-$ in terms of $\mathcal{X}_0$.

In [10] this generalized idea appears in Appendix to IX Example 3, but is not further investigated therein. Moreover, this idea appears in [3, Sec. 2.11] under the name *triplets of spaces*. However, they only scratch the surface as the section is three pages long. In [5] this concept was treated seriously, the authors call it *triplets of closely embedded Hilbert spaces*. The motivation were weighted Sobolev and L^2 spaces, where the positive weight is neither bounded from above nor from below. Independently, [13] also developed this generalization of Gelfand triples under the name *quasi Gelfand triples*, motivated by boundary spaces of differential operators, e.g., the Maxwell operator. This led to a characterization of well-posed boundary conditions for linear (spatially multdimensional) port-Hamiltonian systems. In [6] the authors compare the notions *triplets of spaces* and *triplets of closely embedded Hilbert spaces* and give conditions when they coincide. However, we will show in Appendix A that all of these concepts (*triplets of spaces*, *triplets of closely embedded Hilbert spaces* and *quasi Gelfand triples*) coincide always (no condition needed).

In all previous approaches $\mathcal{X}_+$ was a Hilbert space. Here, amongst others, we lift the setting of [13] to Banach spaces. So the beginning will be relatively similar to the introduction of quasi Gelfand triples in [13]. This lifting has also be done in the Ph.D. thesis [12]. However, we go beyond the refinements of [12] and show that there exists a smallest space where we can structure preservingly embed the entire quasi Gelfand triple in Section 5. Furthermore, we show a bijective relation between quasi Gelfand triples and Gram operators in Section 6. This connection to Gram operators has also been discovered in [5] or it was actually the starting point of their journey. They call the Gram operator *the Hamiltonian* of the triple. However, we take the next step and utilize this connection to the Gram operator to construct a decomposition of the quasi Gelfand triple into two "ordinary" Gelfand triples. These two structural observation make quasi Gelfand triples more accessible in applications. Moreover, they were supposed to solve Conjectures 6.7 and 6.8, but were not enough.

In [4] the authors construct suitable boundary spaces for the tangential trace and the twisted tangential trace that correspond to the curl operator. These spaces naturally form a quasi Gelfand triple with $\mathrm{L}^2(\partial\Omega)$ as pivot space. However, they did not pay a lot of attention to this additional structure as they develop their theory particular for the $\mathrm{H}(\mathrm{curl},\Omega)$ traces (tangential and twisted tangential trace). Moreover, they also give an explicit decomposition of the quasi Gelfand triple into two "ordinary" Gelfand triple (without calling it that).

In Section 3 we will bring up the setting of [4] as a motivation for the notion of quasi Gelfand triple. However, it is also suitable for other pairs of differential operators, e.g., $(\operatorname{symCurl}, \operatorname{Curl})$, $(\operatorname{CurlCurl}^\mathsf{T}, \operatorname{CurlCurl}^\mathsf{T})$, $(\operatorname{symGrad}, \operatorname{Div})$, etc.

There is also a link to the notion of *quasi boundary triples*, which was introduced in [2]. The combination of boundary triples and quasi Gelfand triples is not entirely the same as quasi boundary triples, however both can be used to overcome limitations of boundary triples alone.

2. Preliminary

Since we will often switch between Hilbert space inner products and dual pairings, it is more convenient to always regard the anti-dual space instead of the dual space, which we will do. The anti-dual space is the space of all continuous antilinear mappings from the vector space to $\mathbb{C}$. Moreover, we will use a generalized concept for (unbounded) linear operators, namely linear relations. The following notion of linear relations, dual pairs and adjoints with respect to dual pairs are carefully covered in [12, Ch. 1, Ch. 2]. Linear relations in Hilbert spaces are also properly introduced in [1].

A linear relation T between the vector spaces X and Y is a linear subspace of $X \times Y$. Clearly, every linear operator is also a linear relation (we do not distinguish between a function and its graph). For linear operators we have $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in T$ is equivalent to $Tx = y$. We will use the following notation

$$\ker T := \{x \in X \mid \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in T\}, \qquad \operatorname{ran} T := \{y \in Y \mid \exists x : \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in T\},$$
$$\operatorname{mul} T := \left\{y \in Y \,\middle|\, \left[\begin{smallmatrix} 0 \\ y \end{smallmatrix}\right] \in T\right\}, \qquad \operatorname{dom} T := \{x \in X \mid \exists y : \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in T\}.$$

Thus, T is single-valued (an operator), if $\operatorname{mul} T = \{0\}$. The closure $\overline{T}$ of a linear relation T is the closure in $X \times Y$. Note that every linear relation is closable. Also every operator has a closure as a linear relation, but its closure can be multi-valued. Therefore, showing $\operatorname{mul} \overline{T} = \{0\}$ is necessary, even if $\operatorname{mul} T = \{0\}$.

Definition 2.1. Let X and Y be Banach spaces and let $\langle \cdot, \cdot \rangle_{Y,X} : Y \times X \to \mathbb{C}$ be continuous and sesquilinear (linear in the first argument and antilinear in the second argument). We define

$$\Psi\colon \begin{cases} Y & \to \quad X', \\ y & \mapsto \quad \langle y, \cdot \rangle_{Y,X}, \end{cases} \qquad \text{and} \qquad \Phi\colon \begin{cases} X & \to \quad Y', \\ x & \mapsto \quad \overline{\langle \cdot, x \rangle_{Y,X}}. \end{cases}$$

If Ψ is isometric and bijective, then we say that (X, Y) is an *(anti-)dual pair* and $\langle \cdot, \cdot \rangle_{Y,X}$ is its *(anti-)dual pairing.*

We define

$$\langle x, y \rangle_{X,Y} := \overline{\langle y, x \rangle_{Y,X}},$$

which is again a sesquilinear form.

If also Φ is isometric and bijective, then we say that (X, Y) is a *complete (anti-)dual pair.*

Clearly, (X, X') is a dual pair with the canonical dual pairing $\langle x', x \rangle_{X',X} = x'(x)$ and it is complete, if X is reflexive. For a Hilbert space (H, H) is a complete dual pair with the inner product as dual pairing $\langle x, y \rangle_{H,H} = \langle x, y \rangle_H$. However, if we think of the Sobolev space $\mathrm{H}^1(\mathbb{R})$ there are two "natural" possible dual pairings: the standard Hilbert space (complete) dual pair $(\mathrm{H}^1(\mathbb{R}), \mathrm{H}^1(\mathbb{R}))$ and the dual pair that is induced by the L^2 inner product $(\mathrm{H}^1(\mathbb{R}), \mathrm{H}^{-1}(\mathbb{R}))$ given by $\langle x, y \rangle_{\mathrm{H}^1(\mathbb{R}),\mathrm{H}^{-1}(\mathbb{R})} = \lim_{n\to\infty} \langle x, y_n \rangle_{\mathrm{L}^2(\mathbb{R})}$. Hence, in order to avoid saying both $\mathrm{H}^1(\mathbb{R})$ and $\mathrm{H}^{-1}(\mathbb{R})$ is the dual space of $\mathrm{H}^1(\mathbb{R})$, which can lead to confusion, we prefer the term (complete) dual pair.

Definition 2.2. Let (X_1, Y_1), (X_2, Y_2) be dual pairs and A a linear relation between X_1 and X_2. Then we define the *adjoint linear relation* by

$$A^{*_{Y_2 \times Y_1}} := \left\{ \begin{bmatrix} y_2 \\ y_1 \end{bmatrix} \in Y_2 \times Y_1 \,\middle|\, \langle y_2, x_2 \rangle_{Y_2,X_2} = \langle y_1, x_1 \rangle_{Y_1,X_1} \text{ for all } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in A \right\}.$$

We will just write A^*, if the dual pairs are clear.

For a Banach space X, we will regard the dual pair (X, X') for the adjoint, if no other dual pair is given. Similar, for a Hilbert space H we will regard the dual pair (H, H), if no other dual pair is given.

Note that this definition matches the usual Hilbert space adjoint, if A is a densely defined operator between two Hilbert spaces.

Remark 2.3. If A is an operator ($\operatorname{mul} A = \{0\}$) from X_1 to X_2, then we can characterize the domain of A^* by

$$y_2 \in \operatorname{dom} A^* \;\Leftrightarrow\; \operatorname{dom} A \ni x_1 \mapsto \langle y_2, A x_1 \rangle_{Y_2, X_2} \text{ is continuous w.r.t. } \|\cdot\|_{X_1}.$$

Moreover, we have the following relations

$$\ker A^* = (\operatorname{ran} A)^\perp \quad \text{and} \quad \operatorname{mul} A^* = (\operatorname{dom} A)^\perp,$$

where $M^\perp$ denotes the annihilator space of M (which is the orthogonal complement in the Hilbert space case).

3. Motivation

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded open set with bounded Lipschitz boundary. For $f, g \in \mathrm{C}^\infty(\mathbb{R}^3)$ we have the following integration by parts formula:

$$\langle \operatorname{div} f, g \rangle_{\mathrm{L}^2(\Omega)} + \langle f, \operatorname{grad} g \rangle_{\mathrm{L}^2(\Omega)} = \langle \nu \cdot \gamma_0 f, \gamma_0 g \rangle_{\mathrm{L}^2(\partial\Omega)},$$

where ν is the normal vector on $\partial\Omega$ and γ_0 is the boundary trace. It is also well known that we can extend this formula for $f \in \mathrm{H}(\operatorname{div}, \Omega)$ and $g \in \mathrm{H}^1(\Omega)$:

$$\langle \operatorname{div} f, g \rangle_{\mathrm{L}^2(\Omega)} + \langle f, \operatorname{grad} g \rangle_{\mathrm{L}^2(\Omega)} = \langle \gamma_\nu f, \gamma_0 g \rangle_{\mathrm{H}^{-1/2}(\partial\Omega), \mathrm{H}^{1/2}(\partial\Omega)},$$

where γ_ν is the continuous extension of $\nu \cdot \gamma_0$. In this extension we stumble over the Gelfand triple $(\mathrm{H}^{1/2}(\partial\Omega), \mathrm{L}^2(\partial\Omega), \mathrm{H}^{-1/2}(\partial\Omega))$. However, in general such an integration by parts formula will not automatically lead to such an extension where we can replace the L^2 inner product on the boundary by a dual pairing that comes from a Gelfand triple with $\mathrm{L}^2(\partial\Omega)$ as pivot space. For example for $f, g \in \mathrm{C}^\infty(\mathbb{R}^3)$ we have

$$\langle \operatorname{curl} f, g \rangle_{\mathrm{L}^2(\Omega)} + \langle f, \operatorname{curl} g \rangle_{\mathrm{L}^2(\Omega)} = \langle \nu \times \gamma_0 f, (\nu \times \gamma_0 g) \times \nu \rangle_{\mathrm{L}^2(\partial\Omega)}, \tag{1}$$

but contrary to the previous case neither $\nu \times \gamma_0$ nor $(\nu \times \gamma_0) \times \nu$ can be continuously extended to $\mathrm{H}(\operatorname{curl}, \Omega)$ such that its codomain is still $\mathrm{L}^2(\partial\Omega)$ (or can be continuously embedded into $\mathrm{L}^2(\partial\Omega)$), see [13, Ex. A.4]. Hence, in order to better understand the relation between the extension of (1) to $\mathrm{H}(\operatorname{curl}, \Omega)$ and the $\mathrm{L}^2(\partial\Omega)$ inner product we need a more general tool than Gelfand triples. In order to try to find a suitable boundary space such that we can extend $\nu \times \gamma_0$ on $\mathrm{H}(\operatorname{curl}, \Omega)$, we endow $\operatorname{ran}(\nu \times \gamma_0)$ with the range norm that comes from $\mathrm{H}(\operatorname{curl}, \Omega)$. This gives a norm on a dense subspace of $\mathrm{L}^2_\tau(\partial\Omega) = \{\phi \in \mathrm{L}^2(\partial\Omega) \mid \nu \cdot f = 0\}$ that is unrelated to $\|\cdot\|_{\mathrm{L}^2(\partial\Omega)}$. This setting will be our starting point. This particular problem was treated in [13]. Here we want to discover the world of quasi Gelfand triples without any particular applications in mind (or maybe with Conjectures 6.7 and 6.8 in mind).

Remark 3.1. In [7] the authors stumble over the same problem extending (1). They realized that it has the structure of [10] and used the dual spaces of [4] to extend (1) and construct a boundary triple for the Maxwell operator $\left[\begin{smallmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{smallmatrix}\right]$. Indeed, the dual pairing of their spaces is induced by the pivot space $\mathrm{L}^2_\tau(\partial\Omega)$, but the pivot space is not further used (in its original role) as they apply unitary transformations on the boundary variables to map them into the pivot space, which does not preserve the "natural" relation between the pivot space and the dual pair. This is especially not desirable as it changes the physical meaning of the boundary conditions. Hence, the so constructed boundary triple allows us to parameterize all m-dissipative boundary conditions, but we don't know what their physical meaning is. On the other hand for given (physical) boundary conditions it is rarely possible to find their corresponding parameter to check whether they are m-dissipative. The reason for this is that the unitary transformations are difficult to calculate. Fortunately, the port-Hamiltonian systems in [13] include Maxwell's equations. Hence, it is possible to use boundary triple theory for Maxwell's equations to regard boundary conditions natively in the pivot space $\mathrm{L}^2(\partial\Omega)$, i.e., without unitary transformations, which preserves their physical meaning. This underlines the importance of quasi Gelfand triples.

3.1. Starting point

We will have the following setting: Let $\mathcal{X}_0$ be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{X}_0}$ and $\langle\cdot,\cdot\rangle_{\mathcal{X}_+}$ be another inner product on $\mathcal{X}_0$ (not necessarily related to $\langle\cdot,\cdot\rangle_{\mathcal{X}_0}$), which is defined on a dense (w.r.t. $\|\cdot\|_{\mathcal{X}_0}$) subspace $\tilde{D}_+$ of $\mathcal{X}_0$. We denote the completion of $\tilde{D}_+$ w.r.t. $\|\cdot\|_{\mathcal{X}_+}$ ($\|f\|_{\mathcal{X}_+} := \sqrt{\langle f,f\rangle_{\mathcal{X}_+}}$) by $\mathcal{X}_+$. This completion is, by construction a Hilbert space with the extension of $\langle\cdot,\cdot\rangle_{\mathcal{X}_+}$, for which we use the same symbol. Now we have $\tilde{D}_+$ is dense in $\mathcal{X}_0$ w.r.t. $\|\cdot\|_{\mathcal{X}_0}$ and dense in $\mathcal{X}_+$ w.r.t. $\|\cdot\|_{\mathcal{X}_+}$. Figure 1 illustrates this setting.

Note that $\mathcal{X}_+$, as a Hilbert space, is automatically reflexive. For the further construction the crucial property of $\mathcal{X}_+$ is its reflexivity. Hence, we will weaken the previous setting such that $\mathcal{X}_+$ is only a reflexive Banach space:

- $\mathcal{X}_0$ Hilbert space endowed with $\langle\cdot,\cdot\rangle_{\mathcal{X}_0}$.
- $\tilde{D}_+$ dense subspace of $\mathcal{X}_0$ (w.r.t. $\|\cdot\|_{\mathcal{X}_0}$).
- $\|\cdot\|_{\mathcal{X}_+}$ another norm defined on $\tilde{D}_+$.
- $\mathcal{X}_+$ completion of $\tilde{D}_+$ with respect to $\|\cdot\|_{\mathcal{X}_+}$ is reflexive.

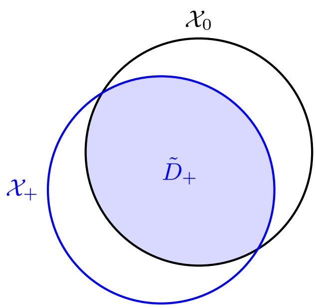


FIGURE 1. Setting of $\mathcal{X}_0$, $\tilde{D}_+$ and $\mathcal{X}_+$.

Example 3.2. Let $\mathcal{X}_0 = \ell^2(\mathbb{Z}\setminus\{0\})$ with the standard inner product $\langle x,y\rangle_{\mathcal{X}_0} = \sum_{n=1}^{\infty} x_n\overline{y_n} + x_{-n}\overline{y_{-n}}$. We define the inner product

$$\langle x,y\rangle_{\mathcal{X}_+} := \sum_{n=1}^{\infty} n^2 x_n\overline{y_n} + \frac{1}{n^2}x_{-n}\overline{y_{-n}}$$

and the set $\tilde{D}_+ := \{f \in \mathcal{X}_0 \,|\, \|f\|_{\mathcal{X}_+} < +\infty\}$. Clearly, this inner product is well-defined on $\tilde{D}_+$. Let e_i denote the sequence which is 1 on the i-th position and 0 elsewhere. Since $\{e_i \,|\, i \in \mathbb{Z}\setminus\{0\}\}$ is a orthonormal basis of $\mathcal{X}_0$ and contained in $\tilde{D}_+$, $\tilde{D}_+$ is dense in $\mathcal{X}_0$. The sequence $\left(\sum_{i=1}^{n} e_{-i}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_{\mathcal{X}_+}$, but not with respect to $\|\cdot\|_{\mathcal{X}_0}$. On the other hand $\left(\sum_{i=1}^{n} \frac{1}{i}e_i\right)_{n\in\mathbb{N}}$ is a Cauchy sequence w.r.t. $\|\cdot\|_{\mathcal{X}_0}$, but not w.r.t. $\|\cdot\|_{\mathcal{X}_+}$.

Definition 3.3. We define

$$\|g\|_{\mathcal{X}_-} := \sup_{f\in\tilde{D}_+\setminus\{0\}} \frac{|\langle g,f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} \quad \text{for } g\in\mathcal{X}_0 \quad \text{and} \quad D_- := \left\{ g\in\mathcal{X}_0 \,\middle|\, \|g\|_{\mathcal{X}_-} < +\infty \right\}.$$

We denote the completion of D_- w.r.t. $\|\cdot\|_{\mathcal{X}_-}$ by $\mathcal{X}_-$. We will also denote the extension of $\|\cdot\|_{\mathcal{X}_-}$ to $\mathcal{X}_-$ by $\|\cdot\|_{\mathcal{X}_-}$.

Remark 3.4. By definition of D_- we can identify every $g \in D_-$ with an element of $\mathcal{X}_+'$ by the continuous extension of

$$\psi_g : \begin{cases} D_+ & \to & \mathbb{C}, \\ f & \mapsto & \langle g, f\rangle_{\mathcal{X}_0}, \end{cases}$$

on $\mathcal{X}_+$. We denote this extension again by ψ_g. By definition of D_- we have $\|\psi_g\|_{\mathcal{X}_+'} = \|g\|_{\mathcal{X}_-}$ for $g \in D_-$. Hence, we can extend the isometry

$$\Psi : \begin{cases} D_- & \to & \mathcal{X}_+', \\ g & \mapsto & \psi_g, \end{cases}$$

by continuity on $\mathcal{X}_-$, this extension is again denoted by Ψ. So $\mathcal{X}_-$ can be seen as the closure of D_- in $\mathcal{X}_+'$.

We can define a dual pairing between $\mathcal{X}_+$ and $\mathcal{X}_-$ by

$$\langle g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} := \langle \Psi g, f\rangle_{\mathcal{X}_+',\mathcal{X}_+}.$$

However, this does not necessarily make $(\mathcal{X}_+, \mathcal{X}_-)$ a dual pair in the sense of Definition 2.1, because we do not know whether Ψ is surjective.

Lemma 3.5. *D_- is complete with respect to* $\|g\|_{\mathcal{X}_-\cap\mathcal{X}_0} := \sqrt{\|g\|_{\mathcal{X}_0}^2 + \|g\|_{\mathcal{X}_-}^2}$.

Proof. Let $(g_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in D_- with respect to $\|\cdot\|_{\mathcal{X}_-\cap\mathcal{X}_0}$. Then $(g_n)_{n\in\mathbb{N}}$ is a convergent sequence in $\mathcal{X}_0$ (w.r.t. $\|\cdot\|_{\mathcal{X}_0}$) and a Cauchy sequence in D_- (w.r.t. $\|\cdot\|_{\mathcal{X}_-}$). We denote the limit in $\mathcal{X}_0$ by g_0. By definition of $\|\cdot\|_{\mathcal{X}_-}$ we obtain for $f \in \tilde{D}_+$

$$|\langle g_0, f\rangle_{\mathcal{X}_0}| = \lim_{n\to\infty} |\langle g_n, f\rangle_{\mathcal{X}_0}| \le \lim_{n\to\infty} \|g_n\|_{\mathcal{X}_-} \|f\|_{\mathcal{X}_+} \le C\|f\|_{\mathcal{X}_+}$$

and consequently $g_0 \in D_-$.

Let $\epsilon > 0$ be arbitrary. Since $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_{\mathcal{X}_-}$, there is an $n_0 \in \mathbb{N}$ such that for all $f \in \tilde{D}_+$ with $\|f\|_{\mathcal{X}_+} = 1$

$$|\langle g_n - g_m, f\rangle_{\mathcal{X}_0}| \le \frac{\epsilon}{2}, \quad \text{if} \quad n, m \ge n_0$$

holds true. Furthermore, for every $f \in \tilde{D}_+$ there exists an $m_f \ge n_0$ such that $|\langle g_0 - g_{m_f}, f\rangle_{\mathcal{X}_0}| \le \frac{\epsilon\|f\|_{\mathcal{X}_+}}{2}$, because $g_m \to g_0$ w.r.t. $\|\cdot\|_{\mathcal{X}_0}$. This yields

$$\frac{|\langle g_0 - g_n, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} \le \frac{|\langle g_0 - g_{m_f}, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} + \frac{|\langle g_{m_f} - g_n, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} \le \epsilon, \quad \text{if} \quad n \ge n_0.$$

Since the right-hand-side is independent of f, we obtain

$$\|g_0 - g_n\|_{\mathcal{X}_-} = \sup_{f\in\tilde{D}_+\setminus\{0\}} \frac{|\langle g_0 - g_n, f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} \le \epsilon, \quad \text{if} \quad n \ge n_0.$$

Hence, g_0 is also the limit of $(g_n)_{n\in\mathbb{N}}$ with respect to $\|\cdot\|_{\mathcal{X}_-}$ and consequently the limit of $(g_n)_{n\in\mathbb{N}}$ with respect to $\|\cdot\|_{\mathcal{X}_-\cap\mathcal{X}_0}$. ❑

Strictly speaking $\tilde{D}_+$ and D_- are subsets of $\mathcal{X}_0$, but sometimes we rather want to regard them as subsets of $\mathcal{X}_+$ and $\mathcal{X}_-$, respectively. Hence, we introduce the following embedding mappings

$$\tilde{\iota}_+ : \begin{cases} \tilde{D}_+ \subseteq \mathcal{X}_+ & \to & \mathcal{X}_0, \\ f & \mapsto & f, \end{cases} \quad \text{and} \quad \iota_- : \begin{cases} D_- \subseteq \mathcal{X}_- & \to & \mathcal{X}_0, \\ g & \mapsto & g. \end{cases}$$

This allows us to distinguish between $f \in \tilde{D}_+$ as element of $\mathcal{X}_+$ and $\tilde{\iota}_+(f)$ as element of $\mathcal{X}_0$, if necessary. Clearly, the same for $g \in D_-$.

Note that there is an asymmetry in the notation, i.e., we use tilde for $\tilde{D}_+$ and $\tilde{\iota}_+$, but no tilde for D_- and ι_-. This is because $\tilde{D}_+$ might be chosen a little bit too small, i.e., $\tilde{D}_+$ is somehow a core of the later introduced D_+. On the other hand D_- is defined maximally.

Lemma 3.6. *The embedding $\tilde{\iota}_+$ is a densely defined operator with $\operatorname{ran} \tilde{\iota}_+$ dense in $\mathcal{X}_0$ and $\ker \tilde{\iota}_+ = \{0\}$. Furthermore, the embedding ι_- is closed and $\ker \iota_- = \{0\}$.*

Proof. By assumption on $\tilde{D}_+$ and definition of $\mathcal{X}_+$ the embedding $\tilde{\iota}_+$ is densely defined and has a dense range. Clearly, $\ker \tilde{\iota}_+ = \{0\}$ and $\ker \iota_- = \{0\}$. By Lemma 3.5 ι_- is closed. ❑

Lemma 3.7. *Let $\tilde{\iota}_+^* = \tilde{\iota}_+^{*\mathcal{X}_0 \times \mathcal{X}_+'}$ denote the adjoint relation (w.r.t. the dualities $(\mathcal{X}_0, \mathcal{X}_0)$ and $(\mathcal{X}_+, \mathcal{X}_+')$) of $\tilde{\iota}_+$. Then $\tilde{\iota}_+^*$ is an operator (single-valued, i.e., $\operatorname{mul} \tilde{\iota}_+^* = \{0\}$) and $\ker \tilde{\iota}_+^* = \{0\}$. Its domain coincides with D_- and $\tilde{\iota}_+^* \iota_- \colon D_- \subseteq \mathcal{X}_- \to \mathcal{X}_+'$ is isometric.*

If $\ker \overline{\tilde{\iota}_+} = \{0\}$, then $\operatorname{ran} \tilde{\iota}_+^*$ is dense in $\mathcal{X}_+'$.

Proof. The density of the domain of $\tilde{\iota}_+$ yields $\operatorname{mul} \tilde{\iota}_+^* = (\operatorname{dom} \tilde{\iota}_+)^\perp = \{0\}$, and $\overline{\operatorname{ran} \tilde{\iota}_+}^{\mathcal{X}_0} = \mathcal{X}_0$ yields $\ker \tilde{\iota}_+^* = \{0\}$. The following equivalences show $\operatorname{dom} \tilde{\iota}_+^* = D_-$:

$$\begin{aligned} g \in \operatorname{dom} \tilde{\iota}_+^* \quad &\Leftrightarrow \quad \tilde{D}_+ \ni f \mapsto \langle g, \tilde{\iota}_+ f \rangle_{\mathcal{X}_0} \text{ is continuous w.r.t. } \|\cdot\|_{\mathcal{X}_+} \\ &\Leftrightarrow \quad \sup_{f \in \tilde{D}_+ \setminus \{0\}} \frac{|\langle g, f \rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} < +\infty \\ &\Leftrightarrow \quad g \in D_-. \end{aligned}$$

For $g \in D_- \subseteq \mathcal{X}_-$ we have

$$\|g\|_{\mathcal{X}_-} = \sup_{f \in \tilde{D}_+ \setminus \{0\}} \frac{|\langle \iota_- g, \tilde{\iota}_+ f \rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} = \sup_{f \in \tilde{D}_+ \setminus \{0\}} \frac{|\langle \tilde{\iota}_+^* \iota_- g, f \rangle_{\mathcal{X}_+', \mathcal{X}_+}|}{\|f\|_{\mathcal{X}_+}} = \|\tilde{\iota}_+^* \iota_- g\|_{\mathcal{X}_+'},$$

which proves that $\tilde{\iota}_+^* \iota_-$ is isometric.

Note that the reflexivity of $\mathcal{X}_+$ implies $\overline{\tilde{\iota}_+} = \tilde{\iota}_+^{**}$. If $\ker \overline{\tilde{\iota}_+} = \{0\}$, then the following equation implies the density of $\operatorname{ran} \tilde{\iota}_+^*$ in $\mathcal{X}_+'$

$$\{0\} = \ker \overline{\tilde{\iota}_+} = \ker \tilde{\iota}_+^{**} = (\operatorname{ran} \tilde{\iota}_+^*)^\perp.$$ ❑

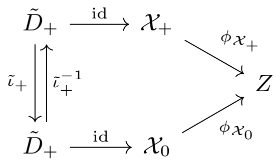
Remark 3.8. As mentioned in Remark 3.4 every $g \in D_-$ can be regarded as an element of $\mathcal{X}_+'$ by ψ_g. Let $g \in D_-$, $f \in \mathcal{X}_+$ and $(f_n)_{n \in \mathbb{N}}$ in $\tilde{D}_+$ converging to f w.r.t. $\|\cdot\|_{\mathcal{X}_+}$. Since $D_- = \operatorname{dom} \tilde{\iota}_+^*$, we have

$$\langle \psi_g, f \rangle_{\mathcal{X}_+', \mathcal{X}_+} = \lim_{n \to \infty} \langle g, f_n \rangle_{\mathcal{X}_0} = \lim_{n \to \infty} \langle \iota_- g, \tilde{\iota}_+ f_n \rangle_{\mathcal{X}_0} = \langle \tilde{\iota}_+^* \iota_- g, f \rangle_{\mathcal{X}_+', \mathcal{X}_+}$$

and consequently $\psi_g = \tilde{\iota}_+^* \iota_- g$. Hence, $\Psi D_- = \tilde{\iota}_+^* \iota_- D_- = \operatorname{ran} \tilde{\iota}_+^*$.

Proposition 3.9. *The following assertions are equivalent.*

(i) *There is a Hausdorff topological vector space* $(Z, \mathcal{T})$ *and two continuous embeddings* $\phi_{\mathcal{X}_+} : \mathcal{X}_+ \to Z$ *and* $\phi_{\mathcal{X}_0} : \mathcal{X}_0 \to Z$ *such that the diagram*



commutes.

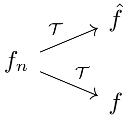
(ii) *If* $\tilde{D}_+ \ni f_n \to 0$ *w.r.t.* $\|\cdot\|_{\mathcal{X}_+}$ *and* $\lim_{n\to\infty} f_n$ *exists w.r.t.* $\|\cdot\|_{\mathcal{X}_0}$*, then this limit is also* 0 *and if* $\tilde{D}_+ \ni f_n \to 0$ *w.r.t.* $\|\cdot\|_{\mathcal{X}_0}$ *and* $\lim_{n\to\infty} f_n$ *exists w.r.t.* $\|\cdot\|_{\mathcal{X}_+}$*, then this limit is also* 0*.*

(iii) $\tilde{\iota}_+ : \tilde{D}_+ \subseteq \mathcal{X}_+ \to \mathcal{X}_0, f \mapsto f$ *is closable (as an operator) and its closure is injective.*

(iv) D_- *is dense in* $\mathcal{X}_0$ *and dense in* $\mathcal{X}_+'$*, i.e.,* ΨD_- *is dense in* $\mathcal{X}_+'$*.*

Proof. We will follow the strategy (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\tilde{D}_+$ such that $f_n \to \hat{f}$ w.r.t. $\mathcal{X}_+$ and $f_n \to f$ w.r.t. $\mathcal{X}_0$. Since $\mathcal{T}$ is coarser than both of the topologies induced by these norms, we also have



in Z.

Since $\mathcal{T}$ is Hausdorff, we conclude $f = \hat{f}$. Hence, if either $\hat{f}$ or f is 0, then also the other is 0.

(ii) $\Rightarrow$ (iii): If $(f_n, f_n)_{n\in\mathbb{N}}$ is a sequence in $\tilde{\iota}_+$ that converges to $(0, f) \in \mathcal{X}_+ \times \mathcal{X}_0$, then $f = 0$ by (ii). Hence, $\operatorname{mul} \overline{\tilde{\iota}_+} = \{0\}$ and consequently $\tilde{\iota}_+$ is closable. On the other hand, if $(f_n, f_n)_{n\in\mathbb{N}}$ is a sequence in $\tilde{\iota}_+$ that converges to $(f, 0)$, then $f = 0$ by (ii). Consequently, $\ker \overline{\tilde{\iota}_+} = \{0\}$ and $\overline{\tilde{\iota}_+}$ is injective.

(iii) $\Rightarrow$ (iv): We have $(\operatorname{dom} \tilde{\iota}_+^*)^\perp = \operatorname{mul} \tilde{\iota}_+^{**} = \operatorname{mul} \overline{\tilde{\iota}_+}$. Since $\tilde{\iota}_+$ is closable, we have $\operatorname{mul} \overline{\tilde{\iota}_+} = \{0\}$, which implies that $\operatorname{dom} \tilde{\iota}_+^*$ is dense in $\mathcal{X}_0$. By Lemma 3.7 $\operatorname{dom} \tilde{\iota}_+^*$ coincides with D_-, which gives the density of D_- in $\mathcal{X}_0$.

The second assertion of Lemma 3.7 yields that $\operatorname{ran} \tilde{\iota}_+^*$ is dense in $\mathcal{X}_+'$. By Remark 3.8 we have $\operatorname{ran} \tilde{\iota}_+^* = \Psi D_-$ and therefore the density of ΨD_- in $\mathcal{X}_+'$.

(iv) $\Rightarrow$ (i): Let $Y := D_-$ be equipped with

$$\|g\|_Y := \|g\|_{\mathcal{X}_- \cap \mathcal{X}_0} = \sqrt{\|g\|_{\mathcal{X}_-}^2 + \|\iota_- g\|_{\mathcal{X}_0}^2}.$$

We define $Z := Y'$ as the (anti-)dual space of Y. Note that $\Psi g = \tilde{\iota}_+^* \iota_- g$ for $g \in D_-$. Then we have

$$|\langle f, \iota_- g\rangle_{\mathcal{X}_0}| \le \|f\|_{\mathcal{X}_0} \|\iota_- g\|_{\mathcal{X}_0} \le \|f\|_{\mathcal{X}_0} \|g\|_Y \qquad \text{for} \quad f \in \mathcal{X}_0, g \in Y$$

and

$$|\langle f, \Psi g\rangle_{\mathcal{X}_+, \mathcal{X}_+'}| \le \|f\|_{\mathcal{X}_+} \underbrace{\|\Psi g\|_{\mathcal{X}_+'}}_{=\|g\|_{\mathcal{X}_-}} \le \|f\|_{\mathcal{X}_+} \|g\|_Y \qquad \text{for} \quad f \in \mathcal{X}_+, g \in Y.$$

Hence, $\phi_{\mathcal{X}_0}\colon f\mapsto\langle f,\iota_-\cdot\rangle_{\mathcal{X}_0}$ and $\phi_{\mathcal{X}_+}\colon f\mapsto\langle f,\Psi\cdot\rangle_{\mathcal{X}_+,\mathcal{X}_+'}$ are continuous mappings from $\mathcal{X}_0$ and $\mathcal{X}_+$, respectively, into Z. The injectivity of these mappings follows from the density of D_- in $\mathcal{X}_0$ and D_- in $\mathcal{X}_+'$ (ΨD_- dense in $\mathcal{X}_+'$), respectively. For $f\in\tilde{D}_+$ we have

$$\phi_{\mathcal{X}_+}f=\langle f,\Psi\cdot\rangle_{\mathcal{X}_+,\mathcal{X}_+'}=\langle f,\tilde{\iota}_+^*\iota_-\cdot\rangle_{\mathcal{X}_+,\mathcal{X}_+'}=\langle\tilde{\iota}_+f,\iota_-\cdot\rangle_{\mathcal{X}_0}=\phi_{\mathcal{X}_0}\circ\tilde{\iota}_+f$$

and consequently the diagram in (i) commutes. ❑

If one and therefore all assertions in Proposition 3.9 are satisfied, then $\mathcal{X}_+\cap\mathcal{X}_0$ is defined as the intersection in Z and complete with the norm $\|\cdot\|_{\mathcal{X}_+\cap\mathcal{X}_0}:=\sqrt{\|\cdot\|_{\mathcal{X}_+}^2+\|\cdot\|_{\mathcal{X}_0}^2}$. Moreover, we define D_+ as the closure of $\tilde{D}_+$ in $\mathcal{X}_+\cap\mathcal{X}_0$ (w.r.t. $\|\cdot\|_{\mathcal{X}_+\cap\mathcal{X}_0}$). Note that although $\mathcal{X}_+\cap\mathcal{X}_0$ may depend on Z, D_+ is independent of Z. We will denote the extension of $\tilde{\iota}_+$ to D_+ by ι_+, which can be expressed by $\iota_+=\overline{\tilde{\iota}_+}$. The adjoint ι_+^* coincides with $\tilde{\iota}_+^*$. Also D_- does not change, if we replace $\tilde{D}_+$ by D_+ in Definition 3.3 and all previous results in this section also hold for D_+ and ι_+ instead of $\tilde{D}_+$ and $\tilde{\iota}_+$, respectively. If $\tilde{\iota}_+$ is already closed, then $D_+=\tilde{D}_+$.

Lemma 3.10. *Let one assertion in Proposition 3.9 be satisfied. Let $Z=Y'$, where $Y=D_-$ endowed with $\|g\|_Y:=\|g\|_{\mathcal{X}_-\cap\mathcal{X}_0}=\sqrt{\|g\|_{\mathcal{X}_-}^2+\|g\|_{\mathcal{X}_0}^2}$ (from Proposition 3.9 (iv) ⇒ (i)). Then we have the following characterization for D_+:*

- $D_+=\operatorname{dom}\iota_-^*$,
- $D_+=\mathcal{X}_+\cap\mathcal{X}_0$ *in* Y'.

Proof. Note that for $g\in D_-$ we have $g=(\iota_+^*)^{-1}\iota_+^*g$ and that $\iota_+^*\iota_-$ is isometric from $D_-=\operatorname{dom}\iota_-\subseteq\mathcal{X}_-$ onto $\operatorname{ran}\iota_+^*=\operatorname{dom}(\iota_+^*)^{-1}\subseteq\mathcal{X}_+'$. The following equivalences show the first assertion:

$$\begin{aligned}f\in\operatorname{dom}\iota_-^*&\Leftrightarrow D_-\ni g\mapsto\langle f,\iota_-g\rangle_{\mathcal{X}_0}\text{ is continuous w.r.t. }\|\cdot\|_{\mathcal{X}_-}\\&\Leftrightarrow D_-\ni g\mapsto\langle f,(\iota_+^*)^{-1}\iota_+^*\iota_-g\rangle_{\mathcal{X}_0}\text{ is continuous w.r.t. }\|\cdot\|_{\mathcal{X}_-}\\&\Leftrightarrow \operatorname{dom}(\iota_+^*)^{-1}\ni h\mapsto\langle f,(\iota_+^*)^{-1}h\rangle_{\mathcal{X}_0}\text{ is continuous w.r.t. }\|\cdot\|_{\mathcal{X}_+'}\\&\Leftrightarrow f\in\operatorname{dom}\big((\iota_+^*)^{-1}\big)^*=\operatorname{dom}\iota_+^{-1}=\operatorname{ran}\iota_+=D_+.\end{aligned}$$

For the second characterization we define $P_+:=\mathcal{X}_+\cap\mathcal{X}_0$ in Y' and we define P_- analogously to D_- in Definition 3.3:

$$\|g\|_{P_-}:=\sup_{f\in P_+\setminus\{0\}}\frac{|\langle g,f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}}\quad\text{and}\quad P_-:=\{g\in\mathcal{X}_0\,|\,\|g\|_{P_-}<+\infty\}.$$

Since $\tilde{D}_+\subseteq P_+$ we have $\|g\|_{\mathcal{X}_-}\le\|g\|_{P_-}$ for $g\in P_-$ and consequently $P_-\subseteq D_-$. Furthermore, we can define $\iota_{P_+}\colon P_+\subseteq\mathcal{X}_+\to\mathcal{X}_0, f\mapsto f$ analogously to $\tilde{\iota}_+$. Note that ι_{P_+} is closed due the completeness of $(\mathcal{X}_+\cap\mathcal{X}_0,\|\cdot\|_{\mathcal{X}_+\cap\mathcal{X}_0})$. Then we have $\operatorname{dom}\iota_{P_+}^*=P_-$ and $\tilde{\iota}_+\subseteq\iota_{P_+}$ and therefore $\iota_{P_+}^*\subseteq\tilde{\iota}_+^*$. For $g\in D_-$ and $f\in P_+$ we have, by definition of $P_+=\mathcal{X}_+\cap\mathcal{X}_0$ in Y',

$$|\langle g,f\rangle_{\mathcal{X}_0}|=|\langle\tilde{\iota}_+^*\iota_-g,f\rangle_{\mathcal{X}_+',\mathcal{X}_+}|\le\|\tilde{\iota}_+^*\iota_-g\|_{\mathcal{X}_+'}\|f\|_{\mathcal{X}_+}=\|g\|_{\mathcal{X}_-}\|f\|_{\mathcal{X}_+},$$

which yields $\|g\|_{P_-}\le\|g\|_{\mathcal{X}_-}$. Hence, $P_-=D_-$, $\iota_{P_+}^*=\tilde{\iota}_+^*$ and $\iota_{P_+}=\overline{\tilde{\iota}_+}$, which is equivalent to $P_+=\mathcal{X}_+\cap\mathcal{X}_0=\overline{\tilde{D}_+}^{\mathcal{X}_+\cap\mathcal{X}_0}=D_+$. ❑

Theorem 3.11. *Let one assertion in Proposition 3.9 be satisfied. Then the continuous extension of $\iota_+^*\iota_-$ denoted by $\overline{\iota_+^*\iota_-}$ equals Ψ. Moreover, Ψ is an isometric isomorphism from $\mathcal{X}_-$ to $\mathcal{X}_-$ and $(\mathcal{X}_+, \mathcal{X}_-)$ is a complete dual pair with*

$$\langle g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} := \langle \Psi g, f\rangle_{\mathcal{X}_+',\mathcal{X}_+}.$$

Note that we have already shown that Ψ is isometric hence it is left to show that it is surjective.

Proof. We have already shown, that $\iota_+^*\iota_- g = \Psi g$ for $g \in D_-$. Since D_- is dense in $\mathcal{X}_-$, we also have $\overline{\iota_+^*\iota_-} g = \Psi g$ for $g \in \mathcal{X}_-$.

If one assertion in Proposition 3.9 is true, then all of them are true. Hence, ΨD_- is dense in $\mathcal{X}_+'$ and because Ψ is isometric $\operatorname{ran}\Psi$ is closed and therefore $\operatorname{ran}\Psi = \mathcal{X}_+'$.

Since Ψ is an isomorphism between $\mathcal{X}_-$ and $\mathcal{X}_+'$, it immediately follows that $(\mathcal{X}_+, \mathcal{X}_-)$ is a complete dual pair with the dual pairing $\langle \cdot, \cdot\rangle_{\mathcal{X}_-,\mathcal{X}_+}$. ❑

Remark 3.12. For $f \in D_+$ and $g \in D_-$ we have

$$\langle g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} = \langle \Psi g, f\rangle_{\mathcal{X}_+',\mathcal{X}_+} = \langle \iota_+^*\iota_- g, f\rangle_{\mathcal{X}_+',\mathcal{X}_+} = \langle \iota_- g, \iota_+ f\rangle_{\mathcal{X}_0} = \langle g, f\rangle_{\mathcal{X}_0}.$$

Since these two sets are dense in $\mathcal{X}_+$ and $\mathcal{X}_-$ respectively, we have for $f \in \mathcal{X}_+$ and $g \in \mathcal{X}_-$

$$\langle g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} = \lim_{(n,m)\to(\infty,\infty)} \langle g_n, f_m\rangle_{\mathcal{X}_0},$$

where $(f_m)_{m\in\mathbb{N}}$ is a sequence in D_+ that converges to f in $\mathcal{X}_+$ and $(g_n)_{n\in\mathbb{N}}$ is a sequence in D_- that converges to g in $\mathcal{X}_-$.

4. Definition and results

The previous section leads to the following definition.

Definition 4.1. Let $(\mathcal{X}_+, \mathcal{X}_-)$ be a complete dual pair and $\mathcal{X}_0$ be a Hilbert space. Furthermore, let $\iota_+ \colon \operatorname{dom}\iota_+ \subseteq \mathcal{X}_+ \to \mathcal{X}_0$ and $\iota_- \colon \operatorname{dom}\iota_- \subseteq \mathcal{X}_- \to \mathcal{X}_0$ be densely defined, closed, and injective linear mappings with dense range. We call $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ a *pre-quasi Gelfand triple*, if

$$\langle g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} = \langle \iota_- g, \iota_+ f\rangle_{\mathcal{X}_0} \tag{2}$$

for all $f \in \operatorname{dom}\iota_+$ and $g \in \operatorname{dom}\iota_-$. The space $\mathcal{X}_0$ will be referred as *pivot space*.

If we additionally have $\operatorname{dom}\iota_+^* = \operatorname{ran}\iota_-$, then we call $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ a *quasi Gelfand triple*.

Figure 2 illustrates the setting of a quasi Gelfand triple. Contrary to the previous section we will regard the adjoint of ι_+ and ι_- with respect to the complete dual pairs $(\mathcal{X}_+, \mathcal{X}_-)$ and $(\mathcal{X}_0, \mathcal{X}_0)$. Therefore, ι_+^* is a densely defined operator from $\mathcal{X}_0$ to $\mathcal{X}_-$ and ι_-^* is a densely defined operator from $\mathcal{X}_0$ to $\mathcal{X}_+$. We could not do this before, because we did not know from the beginning that $(\mathcal{X}_+, \mathcal{X}_-)$ is a complete dual pair.

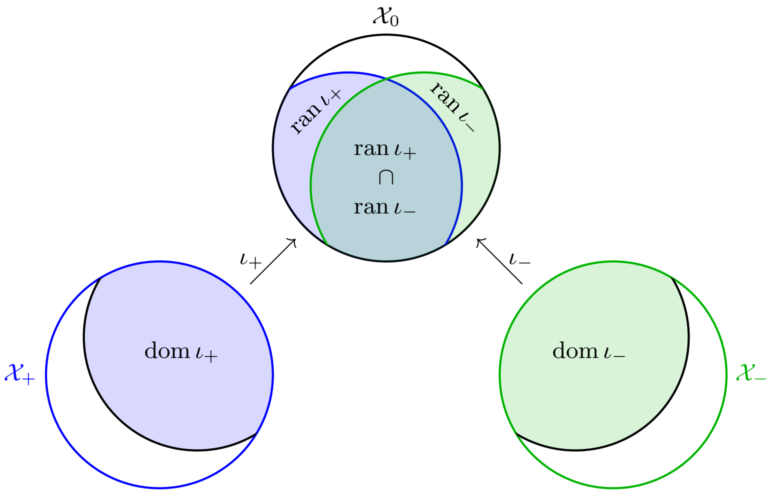


FIGURE 2. Illustration of a quasi Gelfand triple

Example 4.2. Let $\mathcal{X}_+ = \mathrm{L}^p(\mathbb{R})$, $\mathcal{X}_- = \mathrm{L}^q(\mathbb{R})$ and $\mathcal{X}_0 = \mathrm{L}^2(\mathbb{R})$, where $p \in (1, +\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $(\mathcal{X}_+, \mathcal{X}_-)$ is a complete dual pair. Note that $\mathrm{L}^p(\mathbb{R}) \cap \mathrm{L}^2(\mathbb{R})$ is already well-defined. We can define

$$\iota_+ \colon \begin{cases} \mathrm{L}^p(\mathbb{R}) \cap \mathrm{L}^2(\mathbb{R}) \subseteq \mathrm{L}^p(\mathbb{R}) & \to & \mathrm{L}^2(\mathbb{R}), \\ f & \mapsto & f, \end{cases}$$

$$\text{and} \quad \iota_- \colon \begin{cases} \mathrm{L}^q(\mathbb{R}) \cap \mathrm{L}^2(\mathbb{R}) \subseteq \mathrm{L}^q(\mathbb{R}) & \to & \mathrm{L}^2(\mathbb{R}), \\ g & \mapsto & g. \end{cases}$$

These mapping are densely defined, injective and closed with dense range. By definition of the dual pairing of $(\mathrm{L}^p(\mathbb{R}), \mathrm{L}^q(\mathbb{R}))$ we have

$$\langle g, f \rangle_{\mathrm{L}^q(\mathbb{R}), \mathrm{L}^p(\mathbb{R})} = \int_{\mathbb{R}} g\overline{f} \, \mathrm{d}\lambda = \langle g, f \rangle_{\mathcal{X}_0} = \langle \iota_- g, \iota_+ f \rangle_{\mathcal{X}_0}$$

for $g \in \mathrm{L}^q(\mathbb{R}) \cap \mathrm{L}^2(\mathbb{R})$ and $f \in \mathrm{L}^p(\mathbb{R}) \cap \mathrm{L}^2(\mathbb{R})$. By the Hölder inequality we also have $\operatorname{dom} \iota_+^* = \operatorname{ran} \iota_-$. Hence, $(\mathrm{L}^p(\mathbb{R}), \mathrm{L}^2(\mathbb{R}), \mathrm{L}^q(\mathbb{R}))$ is a quasi Gelfand triple.

Note that the mapping ι_+ gives us an identification of $\operatorname{dom} \iota_+$ and $\operatorname{ran} \iota_+$. Hence, we can introduce the norm of $\mathcal{X}_+$ on $\operatorname{ran} \iota_+$ by $\|f\|_{\mathcal{X}_+} = \|\iota_+^{-1} f\|_{\mathcal{X}_+}$ for $f \in \operatorname{ran} \iota_+$. Then the completion of $\operatorname{ran} \iota_+$ with respect to $\|\cdot\|_{\mathcal{X}_+}$ is isometrically isomorphic to $\mathcal{X}_+$. Accordingly, we can do the same for $\mathcal{X}_-$. This justifies the following definition and Figure 3.

Definition 4.3. For a quasi Gelfand triple $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ we define

$$\mathcal{X}_+ \cap \mathcal{X}_0 \coloneqq \operatorname{ran} \iota_+ \quad \text{and} \quad \mathcal{X}_- \cap \mathcal{X}_0 \coloneqq \operatorname{ran} \iota_-.$$

If either ι_+ or ι_- is continuous, then a quasi Gelfand triple is an "ordinary" Gelfand triple. Clearly, every "ordinary" Gelfand triple is also a quasi Gelfand triple.

The additional condition $\operatorname{dom} \iota_+^* = \operatorname{ran} \iota_-$ that makes a pre-quasi Gelfand triple a quasi Gelfand triple is not crucial as it can always be forced, which we will see later in Lemma 4.5. In Conjectures 6.7 and 6.8 we ask ourselves, if this condition is automatically fulfilled. Moreover, the next lemma shows that we can also ask

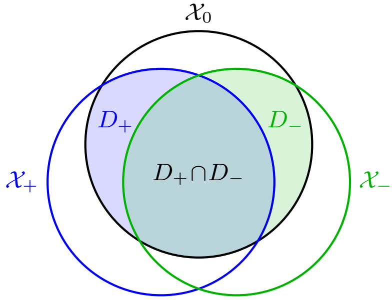


FIGURE 3. Illustration of a quasi Gelfand triple, where $D_+ = \operatorname{ran} \iota_+$ and $D_- = \operatorname{ran} \iota_-$.

for the converse condition $\operatorname{dom} \iota_-^* = \operatorname{ran} \iota_+$ instead. Note that from (2) we can immediately see that $\operatorname{dom} \iota_+^* \supseteq \operatorname{ran} \iota_-$ and $\operatorname{dom} \iota_-^* \supseteq \operatorname{ran} \iota_+$. Hence, for $f \in \operatorname{dom} \iota_+$ and $g \in \operatorname{dom} \iota_-$ we have

$$\langle g, f \rangle_{\mathcal{X}_-, \mathcal{X}_+} = \langle \iota_- g, \iota_+ f \rangle_{\mathcal{X}_0} = \begin{cases} \langle \iota_+^* \iota_- g, f \rangle_{\mathcal{X}_-, \mathcal{X}_+}, \\ \langle g, \iota_-^* \iota_+ f \rangle_{\mathcal{X}_-, \mathcal{X}_+}, \end{cases} \tag{3}$$

which implies $\iota_+^* \iota_- g = g$ and $\iota_-^* \iota_+ f = f$.

Lemma 4.4. *Let $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ be a pre-quasi Gelfand triple with the embeddings ι_+ and ι_-. Then*

$$\operatorname{dom} \iota_+^* = \operatorname{ran} \iota_- \quad \Leftrightarrow \quad \operatorname{dom} \iota_-^* = \operatorname{ran} \iota_+.$$

In particular, if $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a quasi Gelfand triple, then also $\operatorname{dom} \iota_-^* = \operatorname{ran} \iota_+$ *holds true.*

The proof of this is basically the first part of the proof of Lemma 3.10.

Proof. Let $\operatorname{dom} \iota_+^* = \operatorname{ran} \iota_-$. The following equivalences

$$\begin{aligned} f \in \operatorname{dom} \iota_-^* &\Leftrightarrow \operatorname{dom} \iota_- \ni g \mapsto \langle f, \iota_- g \rangle_{\mathcal{X}_0} \text{ is continuous w.r.t. } \|\cdot\|_{\mathcal{X}_-} \\ &\Leftrightarrow \operatorname{dom} \iota_- \ni g \mapsto \langle f, (\iota_+^*)^{-1} \underbrace{\iota_+^* \iota_- g}_{=g} \rangle_{\mathcal{X}_0} \text{ is continuous w.r.t. } \|\cdot\|_{\mathcal{X}_-} \\ &\Leftrightarrow f \in \operatorname{dom} \big((\iota_+^*)^{-1}\big)^* = \operatorname{dom} \iota_+^{-1} = \operatorname{ran} \iota_+ \end{aligned}$$

imply $\operatorname{dom} \iota_-^* = \operatorname{ran} \iota_+$.

The other implication follows analogously. ❑

In contrast to "ordinary" Gelfand triple, the setting for quasi Gelfand triple is somehow "symmetric", i.e., the roles of $\mathcal{X}_+$ and $\mathcal{X}_-$ are interchangeable, since neither of the embeddings ι_+ and ι_- has to be continuous.

Lemma 4.5. *Let $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ be a pre-quasi Gelfand triple with the embeddings ι_+ and ι_-. Then there exists an extension $\hat{\iota}_-$ of ι_- that respects (2) and satisfies* $\operatorname{dom} \iota_+^* = \operatorname{ran} \hat{\iota}_-$. *In particular, $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ with ι_+ and $\hat{\iota}_-$ forms a quasi Gelfand triple.*

Proof. Note that $\iota_+^*\iota_- g = g$. Hence, $\iota_+^* \supseteq \iota_-^{-1}$ and $(\iota_+^*)^{-1} \supseteq \iota_-$. We define $\hat{\iota}_-$ as $(\iota_+^*)^{-1}$. Then clearly $\operatorname{ran}\hat{\iota}_- = \operatorname{dom}\iota_+^*$. For $f \in \operatorname{dom}\iota_+$ and $g \in \operatorname{dom}\hat{\iota}_-$ we have

$$\langle \hat{\iota}_- g, \iota_+ f\rangle_{\mathcal{X}_0} = \langle \iota_+^*\hat{\iota}_- g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} = \langle g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+}. \qquad ❑$$

Alternatively, we could have extended ι_+ by setting $\hat{\iota}_+ := (\iota_-^*)^{-1}$ in the previous lemma to obtain a quasi Gelfand triple.

Remark 4.6. If $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a quasi Gelfand triple and $(\mathcal{X}_+, \widetilde{\mathcal{X}_-})$ is another dual pair for $\mathcal{X}_+$, then also $(\mathcal{X}_+, \mathcal{X}_0, \widetilde{\mathcal{X}_-})$ is a quasi Gelfand triple.

Lemma 4.7. *Let $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ be a quasi Gelfand triple. Then*

$$\iota_+^* = \iota_-^{-1} \quad \textit{and} \quad \iota_-^* = \iota_+^{-1}.$$

Proof. By (3) we have $\iota_+^*\iota_- g = g$ for all $g \in \operatorname{dom}\iota_+$. Since $\operatorname{ran}\iota_- = \operatorname{dom}\iota_+^*$ (by assumption), we conclude that $\iota_+^* = \iota_-^{-1}$.

Analogously, the second equality can be shown. ❑

Theorem 4.8. *Let $\mathcal{X}_+$ be a reflexive Banach space and $\mathcal{X}_0$ be a Hilbert space and $\iota_+ \colon \operatorname{dom}\iota_+ \subseteq \mathcal{X}_+ \to \mathcal{X}_0$ be a densely defined, closed, and injective linear mapping with dense range. Then there exists a Banach space $\mathcal{X}_-$ and a mapping ι_- such that $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a quasi Gelfand triple.*

In particular, $\mathcal{X}_-$ is given by Definition 3.3, where $D_+ = \operatorname{ran}\iota_+$.

Proof. We will identify $\operatorname{dom}\iota_+$ with $\operatorname{ran}\iota_+$ and denote it by D_+. Then item (iii) of Proposition 3.9 is satisfied. Hence, the corresponding D_- (Definition 3.3) is dense in $\mathcal{X}_0$ and its completion $\mathcal{X}_-$ (w.r.t. to $\|\cdot\|_{\mathcal{X}_-}$) establishes the complete dual pair $(\mathcal{X}_+, \mathcal{X}_-)$, by Theorem 3.11. The mapping

$$\iota_- : \begin{cases} D_- \subseteq \mathcal{X}_- & \to & \mathcal{X}_0, \\ \qquad g & \mapsto & g, \end{cases}$$

is densely defined and injective by construction. By the already shown $\operatorname{ran}\iota_- = D_-$ is dense in $\mathcal{X}_0$. Finally, by Lemma 3.6 ι_- is closed and by Lemma 3.7 $\operatorname{dom}\iota_+^* = D_- = \operatorname{ran}\iota_-$. ❑

Remark 4.9. By Theorem 4.8 the setting in the beginning of Section 3 establishes a quasi Gelfand triple, if one assertion of Proposition 3.9 is satisfied.

From now on we will assume that $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a quasi Gelfand triple and we will identify $\operatorname{dom}\iota_+$ with $\operatorname{ran}\iota_+$ and denote it by D_+ as in Figure 3. Analogously, we identify $\operatorname{dom}\iota_-$ with $\operatorname{ran}\iota_-$ and denote it with D_-.

These identifications are really meaningful as we can endow D_+ (as a subset of $\mathcal{X}_0$) with $\|f\|_{\mathcal{X}_+} := \|\iota_+^{-1} f\|_{\mathcal{X}_+}$ for $f \in D_+$. Then the completion of D_+ w.r.t. to this norm is clearly isomorphic to $\mathcal{X}_+$. The same goes for D_-.

The set $D_- = \operatorname{ran}\iota_-$ (previous identification) coincides with the set D_- defined in Definition 3.3 for $\tilde{D}_+ := D_+$.

Proposition 4.10. *The space $D_+ \cap D_-$ is complete with respect to*

$$\|\cdot\|_{\mathcal{X}_+\cap\mathcal{X}_-} := \sqrt{\|\cdot\|_{\mathcal{X}_+}^2 + \|\cdot\|_{\mathcal{X}_-}^2} \quad \textit{and} \quad \|f\|_{\mathcal{X}_0} \leq \|f\|_{\mathcal{X}_+\cap\mathcal{X}_-} \quad \forall f \in D_+ \cap D_-.$$

Proof. For $f \in D_+ \cap D_-$ we have

$$\|f\|_{\mathcal{X}_0}^2 = |\langle f, f\rangle_{\mathcal{X}_0}| = |\langle f, f\rangle_{\mathcal{X}_-,\mathcal{X}_+}| \le \|f\|_{\mathcal{X}_-}\|f\|_{\mathcal{X}_+} \le \|f\|_{\mathcal{X}_+\cap\mathcal{X}_-}^2.$$

Hence, every Cauchy sequence in $D_+ \cap D_-$ with respect to $\|\cdot\|_{\mathcal{X}_+\cap\mathcal{X}_-}$ is also a Cauchy sequence with respect to $\|\cdot\|_{\mathcal{X}_0}$, $\|\cdot\|_{\mathcal{X}_+}$ and $\|\cdot\|_{\mathcal{X}_-}$.

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $D_+ \cap D_-$ with respect to $\|\cdot\|_{\mathcal{X}_+\cap\mathcal{X}_-}$. By the closedness of ι_+ the limit with respect to $\|\cdot\|_{\mathcal{X}_0}$ and the limit with respect to $\|\cdot\|_{\mathcal{X}_+}$ coincide. The same argument for ι_- yields that the limit with respect to $\|\cdot\|_{\mathcal{X}_0}$ and the limit with respect $\|\cdot\|_{\mathcal{X}_-}$ also coincide. Therefore, all these limits have to coincide and $(f_n)_{n\in\mathbb{N}}$ converges to that limit in $D_+ \cap D_-$ w.r.t. $\|\cdot\|_{\mathcal{X}_+\cap\mathcal{X}_-}$. ❑

Lemma 4.11. *The following operator is closed.*

$$\begin{bmatrix}\iota_+ & \iota_-\end{bmatrix} : \begin{cases} D_+ \times D_- \subseteq \mathcal{X}_+ \times \mathcal{X}_- & \to \quad \mathcal{X}_0, \\ \qquad\qquad \begin{bmatrix} f \\ g \end{bmatrix} & \mapsto \quad f + g. \end{cases}$$

Proof. Let $\left(\left(\left[\begin{smallmatrix} f_n \\ g_n \end{smallmatrix}\right], z_n\right)\right)_{n\in\mathbb{N}}$ be a sequence in $\begin{bmatrix}\iota_+ & \iota_-\end{bmatrix}$ that converges to $\left(\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right], z\right)$ in $\mathcal{X}_+ \times \mathcal{X}_- \times \mathcal{X}_0$, i.e.,

$$\begin{aligned} \lim_{n\to\infty} f_n &= f \quad (\text{w.r.t. } \|\cdot\|_{\mathcal{X}_+}), \\ \lim_{n\to\infty} g_n &= g \quad (\text{w.r.t. } \|\cdot\|_{\mathcal{X}_-}), \\ \text{and} \quad \lim_{n\to\infty} f_n + g_n = \lim_{n\to\infty} z_n &= z \quad (\text{w.r.t. } \|\cdot\|_{\mathcal{X}_0}). \end{aligned}$$

Then we have

$$\begin{aligned} \|z_n - z_m\|_{\mathcal{X}_0}^2 &= \|f_n + g_n - (f_m + g_m)\|_{\mathcal{X}_0}^2 = \|(f_n - f_m) + (g_n - g_m)\|_{\mathcal{X}_0}^2 \\ &= \|f_n - f_m\|_{\mathcal{X}_0}^2 + \|g_n - g_m\|_{\mathcal{X}_0}^2 + 2\operatorname{Re}\langle f_n - f_m, g_n - g_m\rangle_{\mathcal{X}_0}. \end{aligned}$$

Note that $\|z_n - z_m\|_{\mathcal{X}_0}^2$ and $\operatorname{Re}\langle f_n - f_m, g_n - g_m\rangle_{\mathcal{X}_0} = \operatorname{Re}\langle f_n - f_m, g_n - g_m\rangle_{\mathcal{X}_+,\mathcal{X}_-}$ converges to 0 for $n, m \to \infty$. Since both $\|f_n - f_m\|_{\mathcal{X}_0}$ and $\|g_n - g_m\|_{\mathcal{X}_0}$ are positive, they also have to converge to 0. Hence, $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are also Cauchy sequences in $\mathcal{X}_0$ and consequently convergent in $\mathcal{X}_0$. By the closedness of ι_+ and ι_- their limits are also f and g, respectively, in $\mathcal{X}_0$. Finally,

$$\begin{bmatrix}\iota_+ & \iota_-\end{bmatrix}\begin{bmatrix} f \\ g \end{bmatrix} = f + g = \lim_{n\to\infty} f_n + \lim_{n\to\infty} g_n = \lim_{n\to\infty} z_n = z$$

finishes the proof. ❑

Proposition 4.12. *$D_+ \cap D_-$ is dense in $\mathcal{X}_0$ with respect to $\|\cdot\|_{\mathcal{X}_0}$.*

Proof. By $\operatorname{dom}\iota_\pm^* = \operatorname{ran}\iota_\mp = D_\mp$ (Lemma 4.4) and $\operatorname{mul}\begin{bmatrix}\iota_+ & \iota_-\end{bmatrix} = \{0\}$ we have

$$\mathcal{X}_0 = \left(\operatorname{mul}\begin{bmatrix}\iota_+ & \iota_-\end{bmatrix}\right)^\perp = \overline{\operatorname{dom}\begin{bmatrix}\iota_+ & \iota_-\end{bmatrix}^*} = \overline{\operatorname{dom}\iota_+^* \cap \operatorname{dom}\iota_-^*} = \overline{D_- \cap D_+}. \quad ❑$$

5. Quasi Gelfand triples with Hilbert spaces

In this section we will regard a quasi Gelfand triple $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$, where $\mathcal{X}_+$ and $\mathcal{X}_-$ (and of course $\mathcal{X}_0$) are Hilbert spaces. Maybe also some of these results can be proven for general quasi Gelfand triple, but we would need a replacement for Theorem B.2.

For a quasi Gelfand triple $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ consisting of Hilbert spaces, there exists a unitary mapping Ψ from $\mathcal{X}_-$ to $\mathcal{X}_+$ (Riesz representation theorem) satisfying

$$\langle g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} = \langle \Psi g, f\rangle_{\mathcal{X}_+} \quad \text{and} \quad \langle f, g\rangle_{\mathcal{X}_+,\mathcal{X}_-} = \langle \Psi^{-1} f, g\rangle_{\mathcal{X}_-}.$$

We will refer to this mapping Ψ as the *duality map* of the quasi Gelfand triple.

Note that we previously regarded the adjoint of ι_+ with respect to the dual pairs $(\mathcal{X}_0, \mathcal{X}_0)$ and $(\mathcal{X}_+, \mathcal{X}_-)$. The main reason for this choice was, that if $\mathcal{X}_+$ is not a Hilbert space, then the dual pair $(\mathcal{X}_+, \mathcal{X}_+)$ is not available, but also sometimes the adjoint with respect to the dual pair $(\mathcal{X}_+, \mathcal{X}_-)$ is more natural.

However, now that $\mathcal{X}_+$ is a Hilbert space, the dual pairs $(\mathcal{X}_+, \mathcal{X}_+)$ and $(\mathcal{X}_-, \mathcal{X}_-)$ are available and seem reasonable when it comes to calculating adjoints. Hence, if we have an additional dual pair (Y, Z) and a linear operator A from $\mathcal{X}_+$ to Y, then we have two choices for the adjoint:

$$A^{*_{Z\times\mathcal{X}_+}}\colon \operatorname{dom} A^* \subseteq Z \to \mathcal{X}_+ \quad \text{and} \quad A^{*_{Z\times\mathcal{X}_-}}\colon \operatorname{dom} A^* \subseteq Z \to \mathcal{X}_-,$$

as defined in Definition 2.2. In order to have a short notation we will denote the adjoints that are taken w.r.t. the dual pairs $(\mathcal{X}_+, \mathcal{X}_+)$ and $(\mathcal{X}_-, \mathcal{X}_-)$ by A^{*_h} (h for Hilbert space duality) and the adjoints w.r.t. $(\mathcal{X}_+, \mathcal{X}_-)$ still by A^*, i.e.,

$$A^{*_\mathrm{h}}\colon \operatorname{dom} A^{*_\mathrm{h}} \subseteq Z \to \mathcal{X}_+ \quad \text{and} \quad A^*\colon \operatorname{dom} A^* \subseteq Z \to \mathcal{X}_-.$$

Clearly, the same goes for mappings, where $\mathcal{X}_+$ is the codomain and analogously for $\mathcal{X}_-$. Note that for $\mathcal{X}_0$ we regard only the dual pair $(\mathcal{X}_0, \mathcal{X}_0)$, therefore we always take adjoints with respect to this dual pair. In particular for ι_+ we have

$$\iota_+^{*_\mathrm{h}}\colon \operatorname{dom} \iota_+^{*_\mathrm{h}} \subseteq \mathcal{X}_0 \to \mathcal{X}_+ \quad \text{and} \quad \iota_+^*\colon \operatorname{dom} \iota_+^* \subseteq \mathcal{X}_0 \to \mathcal{X}_-.$$

By Lemma B.1 we have the following relations between the adjoints:

$$\iota_+^{*_\mathrm{h}} = \Psi \iota_+^* \quad \text{and} \quad \iota_-^{*_\mathrm{h}} = \Psi^{-1} \iota_-^*.$$

Corollary 5.1. *The set $D_+ \cap D_-$ is dense in $\mathcal{X}_+$ and $\mathcal{X}_-$ with respect to their corresponding norms. More precisely* $\operatorname{dom} \iota_+^* \iota_+ = \iota_+^{-1}(D_+ \cap D_-)$ *is dense in $\mathcal{X}_+$ and* $\operatorname{dom} \iota_-^* \iota_- = \iota_-^{-1}(D_+ \cap D_-)$ *is dense in $\mathcal{X}_-$.*

Furthermore, $\iota_+^{-1}(D_+ \cap D_-)$ is a core of ι_+ and $\iota_-^{-1}(D_+ \cap D_-)$ is a core of ι_-.

Proof. Applying Theorem B.2 to ι_+ yields $\iota_+^{*_\mathrm{h}} \iota_+$ is self-adjoint. Note that by Lemma B.1 we have $\iota_+^{*_\mathrm{h}} = \Psi \iota_+^*$, where Ψ is the duality map introduced in the beginning of this section. Hence, $\operatorname{dom} \iota_+^{*_\mathrm{h}} \iota_+ = \operatorname{dom} \iota_+^* \iota_+$ is dense in $\mathcal{X}_+$. By Lemma 4.4 $\operatorname{dom} \iota_+^* = D_-$, consequently

$$\operatorname{dom} \iota_+^* \iota_+ = \iota_+^{-1}(\operatorname{dom} \iota_+^* \cap \operatorname{ran} \iota_+) = \iota_+^{-1}(D_- \cap D_+) = D_+ \cap D_-. \tag{4}$$

Finally, Proposition B.4 and (4) give that $\iota_+^{-1}(D_+ \cap D_-)$ is a core of ι_+.

An analogous argument for ι_- yields $D_+ \cap D_-$ is dense in $\mathcal{X}_-$. ❑

Corollary 5.2. *$D_+ + D_- = \mathcal{X}_0$.*

Proof. Applying Theorem B.2 to ι_+ gives that $(\mathrm{I}_{\mathcal{X}_0} + \iota_+ \iota_+^{*_\mathrm{h}})$ is onto. Hence, for every $x \in \mathcal{X}_0$ there exists a $g_x \in \operatorname{dom} \iota_+ \iota_+^{*_\mathrm{h}} \subseteq D_-$ such that

$$x = \underbrace{g_x}_{\in D_-} + \underbrace{\iota_+ \iota_+^{*_\mathrm{h}} g_x}_{\in D_+}.$$

Since $g_x \in \operatorname{dom} \iota_+ \iota_+^{*_\mathrm{h}}$, we have $\iota_+^{*_\mathrm{h}} g_x \in D_+$ and consequently $x \in D_+ + D_-$. ❑

Next we will show that we can embed an entire quasi Gelfand triple structure preservingly into a larger space. We will even give the smallest possible space that contains the entire quasi Gelfand triple. However, before we start we give a proper definition of what we mean.

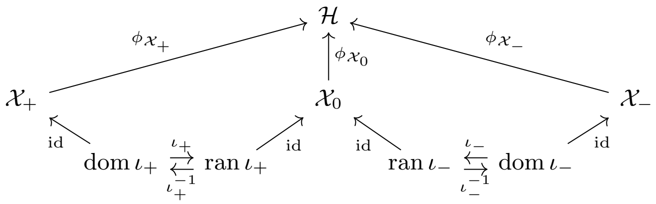
Definition 5.3. Let $\mathcal{H}$ be a Hausdorff topological vector space. We say the quasi Gelfand triple $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ can be *structure preservingly embedded* into $\mathcal{H}$, if there exist linear, injective and continuous mappings

$$\phi_{\mathcal{X}_+} \colon \mathcal{X}_+ \to \mathcal{H}, \quad \phi_{\mathcal{X}_0} \colon \mathcal{X}_0 \to \mathcal{H} \quad \text{and} \quad \phi_{\mathcal{X}_-} \colon \mathcal{X}_- \to \mathcal{H}$$

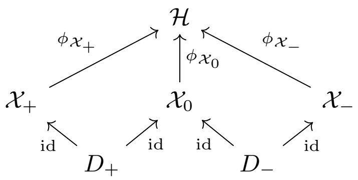
such that

$$\phi_{\mathcal{X}_+}\big|_{\operatorname{dom}\iota_+} = \phi_{\mathcal{X}_0}\iota_+ \quad \text{and} \quad \phi_{\mathcal{X}_-}\big|_{\operatorname{dom}\iota_-} = \phi_{\mathcal{X}_0}\iota_-. \tag{5}$$

Basically the previous definition means that the following diagram commutes.



Since we identify $\operatorname{dom}\iota_+$ and $\operatorname{ran}\iota_+$ with each other and denote it as D_+ and the same for ι_-, we can reduce the previous diagram to the following diagram.



From this point of view the compatibility condition (5) can be seen as

$$\phi_{\mathcal{X}_+} f = \phi_{\mathcal{X}_0} f \quad \forall f \in D_+ \quad \text{and} \quad \phi_{\mathcal{X}_-} g = \phi_{\mathcal{X}_0} g \quad \forall g \in D_-.$$

Note if $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is an "ordinary" Gelfand triple (where ι_+ is continuous), then it is usually denoted by $\mathcal{X}_+ \subseteq \mathcal{X}_0 \subseteq \mathcal{X}_-$. To be precise these inclusions are actually identifications via the mappings ι_+ and ι_-^{-1}. The continuity and closedness of ι_+ implies $\operatorname{dom}\iota_+ = \mathcal{X}_+$ and that ι_+^* is also continuous and everywhere defined. Since $\iota_+^* = \iota_-^{-1}$ (Lemma 4.7), we have the following setting:

$$\mathcal{X}_+ \xrightarrow{\iota_+} \mathcal{X}_0 \xrightarrow{\iota_-^{-1}} \mathcal{X}_-,$$

which suggests that $\mathcal{X}_-$ contains the entire Gelfand triple. Defining $\phi_{\mathcal{X}_+} = \iota_-^{-1}\iota_+$, $\phi_{\mathcal{X}_0} = \iota_-^{-1}$ and $\phi_{\mathcal{X}_-} = \mathrm{id}_{\mathcal{X}_-}$ justifies that $\mathcal{X}_-$ contains the Gelfand triple in a structure preserving manner as defined in Definition 5.3.

For quasi Gelfand triples the construction of a space that covers the entire quasi Gelfand triple needs a bit more attention.

By Proposition 4.10, $D_+ \cap D_-$ with $\|\cdot\|_{\mathcal{X}_+ \cap \mathcal{X}_-}$ is complete and therefore a Banach space. Since $\mathcal{X}_+$ and $\mathcal{X}_-$ are Hilbert spaces (in this section) we can define the inner product

$$\langle g, f\rangle_{\mathcal{X}_+ \cap \mathcal{X}_-} \coloneqq \langle g, f\rangle_{\mathcal{X}_+} + \langle g, f\rangle_{\mathcal{X}_-}$$

on $D_+ \cap D_-$. This inner product induces the previous norm $\|\cdot\|_{\mathcal{X}_+\cap\mathcal{X}_-}$. Consequently $D_+ \cap D_-$ is a Hilbert space with $\langle\cdot,\cdot\rangle_{\mathcal{X}_+\cap\mathcal{X}_-}$. For shorter notation we denote $D_+ \cap D_-$ by $\mathcal{Z}_+$, the corresponding inner product and norm by $\langle\cdot,\cdot\rangle_{\mathcal{Z}_+}$ and $\|\cdot\|_{\mathcal{Z}_+}$, respectively.

Corollary 5.4. *Let $\mathcal{Z}_+ = D_+ \cap D_-$ be the space defined in the previous paragraph. Then the triple $(\mathcal{Z}_+, \mathcal{X}_0, \mathcal{Z}_+')$ forms an "ordinary" Gelfand triple. In particular $\mathcal{Z}_+'$ is isometrically isomorphic to $\mathcal{Z}_-$, the completion of $\mathcal{X}_0$ w.r.t.*

$$\|h\|_{\mathcal{Z}_-} := \sup_{z\in\mathcal{Z}_+\setminus\{0\}} \frac{|\langle h, z\rangle_{\mathcal{X}_0}|}{\|z\|_{\mathcal{Z}_+}}.$$

Proof. By Proposition 4.12 we know that $\mathcal{Z}_+$ is dense in $\mathcal{X}_0$ and by Proposition 4.10 that the mapping $\iota_{\mathcal{Z}_+} : \mathcal{Z}_+ \to \mathcal{X}_0$, $z \mapsto z$ is continuous. Hence, "ordinary" Gelfand triple theory or Theorem 4.8 gives the assertion. ❑

Theorem 5.5. *We can structure preservingly embed the quasi Gelfand triple $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ into the space $\mathcal{Z}_+'$ by the embeddings*

$$\psi_{\mathcal{X}_+} f = \langle f, \iota_-^{-1}\cdot\rangle_{\mathcal{X}_+,\mathcal{X}_-}, \quad \psi_{\mathcal{X}_0} h = \langle h, \cdot\rangle_{\mathcal{X}_0} \quad \textit{and} \quad \psi_{\mathcal{X}_-} g = \langle g, \iota_+^{-1}\cdot\rangle_{\mathcal{X}_-,\mathcal{X}_+}.$$

Note that by our identifications of D_+ and D_- we have $\iota_+^{-1} z = z$ and $\iota_-^{-1} z = z$ for $z \in \mathcal{Z}_+$. However, making this change of spaces visible can sometimes help. Nevertheless, most of the time this is only additional dead weight, this is why we will often just write $\psi_{\mathcal{X}_+}(f)(z) = \langle f, z\rangle_{\mathcal{X}_+,\mathcal{X}_-}$, etc.

Clearly, since $\mathcal{Z}_+'$ and $\mathcal{Z}_-$ are isometrically isomorphic we can also structure preservingly embed $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ into $\mathcal{Z}_-$. For notational harmony we prefer to use $\mathcal{Z}_-$ instead of $\mathcal{Z}_+'$. However, for our purpose there is no need to strictly distinguish between them, this is why we will use these symbols as synonyms. Figure 4 illustrates the meaning of the previous theorem.

Proof. First we have to check that these mappings are well-defined: Let $z \in \mathcal{Z}_+$, $f \in \mathcal{X}_+$, $h \in \mathcal{X}_0$ and $g \in \mathcal{X}_-$. Then

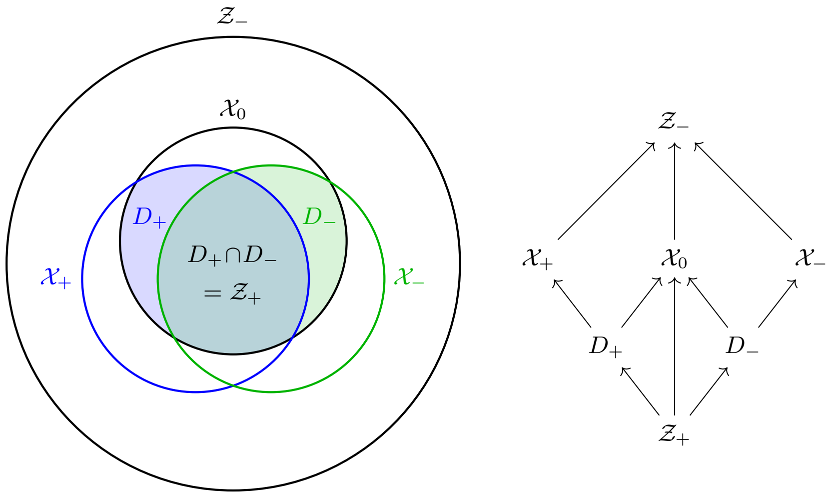
$$\begin{aligned}
|\psi_{\mathcal{X}_+}(f)(z)| &= |\langle f, z\rangle_{\mathcal{X}_+,\mathcal{X}_-}| \le \|f\|_{\mathcal{X}_+}\|z\|_{\mathcal{X}_-} \le \|f\|_{\mathcal{X}_+}\|z\|_{\mathcal{Z}_+},\\
|\psi_{\mathcal{X}_0}(h)(z)| &= |\langle h, z\rangle_{\mathcal{X}_0}| \le \|h\|_{\mathcal{X}_0}\|z\|_{\mathcal{X}_0} \le \|h\|_{\mathcal{X}_0}\|z\|_{\mathcal{Z}_+},\\
|\psi_{\mathcal{X}_-}(g)(z)| &= |\langle g, z\rangle_{\mathcal{X}_-,\mathcal{X}_+}| \le \|g\|_{\mathcal{X}_-}\|z\|_{\mathcal{X}_+} \le \|g\|_{\mathcal{X}_-}\|z\|_{\mathcal{Z}_+},
\end{aligned}$$

which implies $\psi_{\mathcal{X}_+}(f)$, $\psi_{\mathcal{X}_0}(h)$ and $\psi_{\mathcal{X}_-}(g)$ are in $\mathcal{Z}_+'$, and $\psi_{\mathcal{X}_+}$, $\psi_{\mathcal{X}_0}$ and $\psi_{\mathcal{X}_-}$ are continuous. The linearity of $\psi_{\mathcal{X}_+}$, $\psi_{\mathcal{X}_0}$ and $\psi_{\mathcal{X}_-}$ follows from the sesquilinearity of a dual pairing. If $\psi_{\mathcal{X}_+}(f) = 0$, then $f \perp \iota_-^{-1}\mathcal{Z}_+ = \iota_-^{-1}(D_+ \cap D_-) = \operatorname{dom} \iota_-^*\iota_-$. Since $\operatorname{dom} \iota_-^*\iota_-$ is dense in $\mathcal{X}_-$, we conclude $f = 0$, which proves $\phi_{\mathcal{X}_+}$ is injective. Analogously, we can show that $\psi_{\mathcal{X}_-}$ is injective. If $\psi_{\mathcal{X}_0}(h) = 0$, then $h \perp \mathcal{Z}_+$. Since $\mathcal{Z}_+$ is dense in $\mathcal{X}_0$, h has to be 0, which gives the injectivity of $\psi_{\mathcal{X}_0}$. The compatibility condition (5) follows from

$$\begin{aligned}
\psi_{\mathcal{X}_0} \circ \iota_+(f)(z) &= \langle \iota_+ f, z\rangle_{\mathcal{X}_0} = \langle f, \iota_+^* z\rangle_{\mathcal{X}_+,\mathcal{X}_-} = \langle f, \iota_-^{-1} z\rangle_{\mathcal{X}_+,\mathcal{X}_-} = \psi_{\mathcal{X}_+}(f)(z),\\
\psi_{\mathcal{X}_0} \circ \iota_-(g)(z) &= \langle \iota_- g, z\rangle_{\mathcal{X}_0} = \langle g, \iota_-^* z\rangle_{\mathcal{X}_-,\mathcal{X}_+} = \langle g, \iota_+^{-1} z\rangle_{\mathcal{X}_-,\mathcal{X}_+} = \psi_{\mathcal{X}_-}(g)(z),
\end{aligned}$$

which finishes the proof. ❑

Now since we can always structure preservingly embed a quasi Gelfand triple into $\mathcal{Z}_-$ ($\mathcal{Z}_+'$) we can regard this quasi Gelfand triple as subsets of $\mathcal{Z}_-$, see Figure 4a, and do not have to deal with all this embeddings (most of the time). However, we will not get completely rid of these embeddings, as they are sometimes helpful, but we can always regard them as identity mappings.



(A) Venn diagram (B) Commutative diagram

FIGURE 4. quasi Gelfand triple embedded in $\mathcal{Z}_-$

Lemma 5.6. *$\mathcal{Z}_- = \mathcal{X}_+ + \mathcal{X}_-$ and*

$$\|h\|_{\mathcal{Z}_-} = \inf_{\substack{f+g=h \\ f\in\mathcal{X}_+, g\in\mathcal{X}_-}} \sqrt{\|f\|_{\mathcal{X}_+}^2 + \|g\|_{\mathcal{X}_-}^2}.$$

Proof. Note that $\mathcal{Z}_+$ is a Hilbert space with $\langle z_1, z_2\rangle_{\mathcal{Z}_+} = \langle z_1, z_2\rangle_{\mathcal{X}_+} + \langle z_1, z_2\rangle_{\mathcal{X}_-}$. Hence, there is a duality map Φ from $\mathcal{Z}_-$ to $\mathcal{Z}_+$ and we can write

$$\langle h, z\rangle_{\mathcal{Z}_-,\mathcal{Z}_+} = \langle \Phi h, z\rangle_{\mathcal{Z}_+} = \langle \Phi h, z\rangle_{\mathcal{X}_+} + \langle \Phi h, z\rangle_{\mathcal{X}_-}.$$

Furthermore, with the duality map Ψ from $\mathcal{X}_-$ to $\mathcal{X}_+$ we have

$$\langle h, z\rangle_{\mathcal{Z}_-,\mathcal{Z}_+} = \langle \Psi^{-1}\Phi h, z\rangle_{\mathcal{X}_-,\mathcal{X}_+} + \langle \Psi\Phi h, z\rangle_{\mathcal{X}_+,\mathcal{X}_-}$$

and $h = \Psi^{-1}\Phi h + \Psi\Phi h$ in $\mathcal{Z}_-$, where $\Psi^{-1}\Phi h \in \mathcal{X}_-$ and $\Psi\Phi h \in \mathcal{X}_+$.

Let $h \in \mathcal{Z}_-$. Then for every $f \in \mathcal{X}_+$, $g \in \mathcal{X}_-$ that satisfy $h = f + g$ in $\mathcal{Z}_-$ we have

$$\begin{aligned}
|\langle h, z\rangle_{\mathcal{Z}_-,\mathcal{Z}_+}| &= |\langle f, z\rangle_{\mathcal{X}_+,\mathcal{X}_-} + \langle g, z\rangle_{\mathcal{X}_-,\mathcal{X}_+}| \leq |\langle f, z\rangle_{\mathcal{X}_+,\mathcal{X}_-}| + |\langle g, z\rangle_{\mathcal{X}_-,\mathcal{X}_+}| \\
&\leq \|f\|_{\mathcal{X}_+}\|z\|_{\mathcal{X}_-} + \|g\|_{\mathcal{X}_-}\|z\|_{\mathcal{X}_+} \\
&\leq \sqrt{\|f\|_{\mathcal{X}_+}^2 + \|g\|_{\mathcal{X}_-}^2}\sqrt{\|z\|_{\mathcal{X}_-}^2 + \|z\|_{\mathcal{X}_+}^2} \\
&= \sqrt{\|f\|_{\mathcal{X}_+}^2 + \|g\|_{\mathcal{X}_-}^2}\,\|z\|_{\mathcal{Z}_+},
\end{aligned}$$

which implies $\|h\|_{\mathcal{Z}_-} \leq \inf_{h=f+g} \sqrt{\|f\|_{\mathcal{X}_+}^2 + \|g\|_{\mathcal{X}_-}^2}$. On the other hand

$$\|h\|_{\mathcal{Z}_-}^2 = \|\Phi h\|_{\mathcal{Z}_+}^2 = \|\Phi h\|_{\mathcal{X}_+}^2 + \|\Phi h\|_{\mathcal{X}_-}^2 = \|\Psi^{-1}\Phi h\|_{\mathcal{X}_-}^2 + \|\Psi\Phi h\|_{\mathcal{X}_+}^2$$

finishes the proof. ❑

The next result reinforces Definition 4.3.

Proposition 5.7. *The intersection* $\mathcal{X}_+ \cap \mathcal{X}_0$ *in* $\mathcal{Z}_-$ *equals* D_+, *i.e.,* $\operatorname{ran}\psi_{\mathcal{X}_+} \cap \operatorname{ran}\psi_{\mathcal{X}_0} = \operatorname{ran}(\psi_{\mathcal{X}_0} \circ \iota_+)$, *and the intersection* $\mathcal{X}_- \cap \mathcal{X}_0$ *in* $\mathcal{Z}_-$ *equals* D_-, *i.e.,* $\operatorname{ran}\psi_{\mathcal{X}_-} \cap \operatorname{ran}\psi_{\mathcal{X}_0} = \operatorname{ran}(\psi_{\mathcal{X}_0} \circ \iota_-)$.

Proof. Let $h \in \mathcal{X}_+ \cap \mathcal{X}_0 \subseteq \mathcal{Z}_-$, i.e., there exists an $f \in \mathcal{X}_+$ and a $k \in \mathcal{X}_0$ such that

$$\langle h, z\rangle_{\mathcal{Z}_-,\mathcal{Z}_+} = \langle f, \iota_-^{-1} z\rangle_{\mathcal{X}_+,\mathcal{X}_-} = \langle k, z\rangle_{\mathcal{X}_0} \quad \text{for all} \quad z \in \mathcal{Z}_+ = D_+ \cap D_-.$$

We define $x = \iota_-^{-1} z$, which leads to

$$\langle f, x\rangle_{\mathcal{X}_+,\mathcal{X}_-} = \langle k, \iota_- x\rangle_{\mathcal{X}_0} \quad \text{for all} \quad x \in \iota_-^{-1}(D_+ \cap D_-).$$

Since $\iota_-^{-1}(D_+ \cap D_-)$ is a core of ι_- (Corollary 5.1), this equation is also true for all $x \in \operatorname{dom}\iota_-$. Moreover, this implies $f = \iota_-^* k$ and $k \in \operatorname{dom}\iota_-^* = D_+$. By $\iota_-^* = \iota_+^{-1}$ we obtain $\iota_+ f = k \in D_+$ and

$$\langle h, z\rangle_{\mathcal{Z}_-,\mathcal{Z}_+} = \langle f, \iota_-^{-1} z\rangle_{\mathcal{X}_+,\mathcal{X}_-} = \langle k, z\rangle_{\mathcal{X}_0} = \langle \iota_+ f, z\rangle_{\mathcal{X}_0},$$

which gives $h = f = k = \iota_+ f$ in $\mathcal{Z}_-$ and $h \in D_+$.

The same steps can also be done for $\mathcal{X}_-$. ❑

Theorem 5.8. *The intersection* $\mathcal{X}_+ \cap \mathcal{X}_-$ *in* $\mathcal{Z}_-$ *is* $D_+ \cap D_-(= \mathcal{Z}_+)$, *i.e.,*

$$\operatorname{ran}\psi_{\mathcal{X}_+} \cap \operatorname{ran}\psi_{\mathcal{X}_-} = \operatorname{ran}(\psi_{\mathcal{X}_0} \circ \iota_+) \cap \operatorname{ran}(\psi_{\mathcal{X}_0} \circ \iota_-) = \psi_{\mathcal{X}_0}(\mathcal{Z}_+).$$

This means that area of $\mathcal{X}_+ \cap \mathcal{X}_-$ in Figure 4 outside of $\mathcal{X}_0$ is actually empty.

Proof. Let $h \in \mathcal{X}_+ \cap \mathcal{X}_- \subseteq \mathcal{Z}_-$, i.e., there exists an $f \in \mathcal{X}_+$ and a $g \in \mathcal{X}_-$ such that

$$\langle h, z\rangle_{\mathcal{Z}_-,\mathcal{Z}_+} = \langle f, \iota_-^{-1} z\rangle_{\mathcal{X}_+,\mathcal{X}_-} = \langle g, \iota_+^{-1} z\rangle_{\mathcal{X}_-,\mathcal{X}_+} \quad \text{for all} \quad z \in D_+ \cap D_-.$$

We define $x := \iota_+^{-1} z$, which leads to $z = \iota_+ x$. Since $z \in \operatorname{dom}\iota_-^{-1}$, we have $x \in \operatorname{dom}\iota_-^{-1}\iota_+$. Recall that $\iota_-^{-1} = \iota_+^*$ and $\iota_+^{-1}\mathcal{Z}_+ = \iota_+^{-1}(D_+ \cap D_-) = \operatorname{dom}\iota_+^*\iota_+$ (see Lemma 4.7 and Corollary 5.1). Hence,

$$\langle f, \iota_+^*\iota_+ x\rangle_{\mathcal{X}_+,\mathcal{X}_-} = \langle g, x\rangle_{\mathcal{X}_-,\mathcal{X}_+} \quad \text{for all} \quad x \in \operatorname{dom}\iota_+^*\iota_+,$$

which implies $(\iota_+^*\iota_+)^* f = g$ and $f \in \operatorname{dom}(\iota_+^*\iota_+)^*$. By Proposition B.4 $(\iota_+^*\iota_+)^* = \iota_+^*\iota_+$ and therefore $f \in \operatorname{dom}\iota_+^*\iota_+$ and in particular, $\iota_+ f \in \iota_+(\operatorname{dom}\iota_+^*\iota_+) = D_+ \cap D_-$. Note that again by $\iota_-^{-1} = \iota_+^*$ we have $\iota_-^{-1}\iota_+ f = g$. Thus, $g \in \operatorname{dom}\iota_-$ and $\iota_+ f = \iota_- g$. This gives

$$\langle h, z\rangle_{\mathcal{Z}_-,\mathcal{Z}_+} = \langle \iota_+ f, z\rangle_{\mathcal{X}_0} = \langle \iota_- g, z\rangle_{\mathcal{X}_0}.$$

Therefore, $h = f = g = \iota_+ f = \iota_- g$ in $\mathcal{Z}_-$. ❑

Corollary 5.9. *For* $f \in \mathcal{X}_+$ *and* $g \in \mathcal{X}_-$ *we have*

$$\|f + g\|_{\mathcal{Z}_-} = \inf_{z \in \mathcal{Z}_+} \sqrt{\|f + z\|_{\mathcal{X}_+}^2 + \|g - z\|_{\mathcal{X}_-}^2}.$$

Proof. By Lemma 5.6 we have

$$\|f + g\|_{\mathcal{Z}_-} = \inf_{\substack{\tilde{f}+\tilde{g}=f+g \\ \tilde{f}\in\mathcal{X}_+,\tilde{g}\in\mathcal{X}_-}} \sqrt{\|\tilde{f}\|_{\mathcal{X}_+}^2 + \|\tilde{g}\|_{\mathcal{X}_-}^2}$$

Note that $f + g = \tilde{f} + \tilde{g}$ implies

$$z := \underbrace{f - \tilde{f}}_{\in\mathcal{X}_+} = -\underbrace{(g - \tilde{g})}_{\in\mathcal{X}_-} \in \mathcal{X}_+ \cap \mathcal{X}_-.$$

We can write $\tilde{f} = f - z$ and $\tilde{g} = f + z$ and by Theorem 5.8 we have $z \in \mathcal{Z}_+$. Consequently,

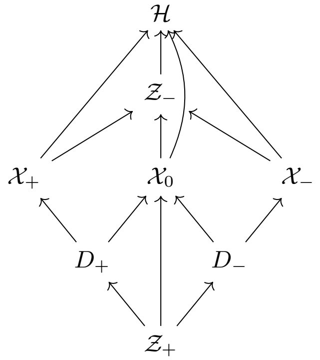
$$\|f+g\| = \inf_{z\in\mathcal{Z}_+} \sqrt{\|f-z\|_{\mathcal{X}_+}^2 + \|g+z\|_{\mathcal{X}_-}^2} = \inf_{z\in\mathcal{Z}_+} \sqrt{\|f+z\|_{\mathcal{X}_+}^2 + \|g-z\|_{\mathcal{X}_-}^2}. \quad \square$$

The space $\mathcal{Z}_-$ is the smallest space where we can embed the quasi Gelfand triple structure preservingly. The following theorem makes this statement precise.

Theorem 5.10. *Let $\mathcal{H}$ be a Hausdorff topological vector space such that we can structure preservingly embed the quasi Gelfand triple $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ into $\mathcal{H}$ and let $\phi_{\mathcal{X}_+}$, $\phi_{\mathcal{X}_0}$ and $\phi_{\mathcal{X}_-}$ denote the embeddings. Then also $\mathcal{Z}_-$ can be continuously embedded into $\mathcal{H}$ by a mapping $\phi_{\mathcal{Z}_-}$, such that*

$$\phi_{\mathcal{Z}_-} \circ \psi_{\mathcal{X}_+} = \phi_{\mathcal{X}_+}, \quad \phi_{\mathcal{Z}_-} \circ \psi_{\mathcal{X}_0} = \phi_{\mathcal{X}_0} \quad \text{and} \quad \phi_{\mathcal{Z}_-} \circ \psi_{\mathcal{X}_-} = \phi_{\mathcal{X}_-},$$

i.e., the following diagram commutes.



Proof. Recall that we can assume that $\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_- \subseteq \mathcal{Z}_-$ and $\psi_{\mathcal{X}_+} f = f$, $\psi_{\mathcal{X}_0} h = h$, and $\psi_{\mathcal{X}_-} g = g$, by simply replacing the quasi Gelfand triple $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ by $(\psi_{\mathcal{X}_+}(\mathcal{X}_+), \psi_{\mathcal{X}_0}(\mathcal{X}_0), \psi_{\mathcal{X}_-}(\mathcal{X}_-))$, see Figure 4.

For convenience we define $\hat{\mathcal{X}}_+ = \phi_{\mathcal{X}_+}(\mathcal{X}_+)$, $\hat{\mathcal{X}}_0 = \phi_{\mathcal{X}_0}(\mathcal{X}_0)$ and $\hat{\mathcal{X}}_- = \phi_{\mathcal{X}_-}(\mathcal{X}_-)$ with the norms $\|f\|_{\hat{\mathcal{X}}_+} = \|\phi_{\mathcal{X}_+}^{-1} f\|_{\mathcal{X}_+}$, $\|h\|_{\hat{\mathcal{X}}_0} = \|\phi_{\mathcal{X}_0}^{-1} h\|_{\mathcal{X}_0}$ and $\|g\|_{\hat{\mathcal{X}}_-} = \|\phi_{\mathcal{X}_-}^{-1} g\|_{\mathcal{X}_-}$.

We will show as a first step that we can endow $\hat{\mathcal{X}}_+ + \hat{\mathcal{X}}_-$ in $\mathcal{H}$ with $\|h\|_{\hat{\mathcal{X}}_+ + \hat{\mathcal{X}}_-} = \inf_{f+g=h} \sqrt{\|f\|_{\hat{\mathcal{X}}_+}^2 + \|g\|_{\hat{\mathcal{X}}_-}^2}$ such that the corresponding topology of $\|\cdot\|_{\hat{\mathcal{X}}_+ + \hat{\mathcal{X}}_-}$ is finer than the topology $\mathcal{T}_{\mathcal{H}}$ of $\mathcal{H}$ (i.e., whenever $(h_n)_{n\in\mathbb{N}}$ converges w.r.t. $\|\cdot\|_{\hat{\mathcal{X}}_+ + \hat{\mathcal{X}}_-}$, it also converges w.r.t. $\mathcal{T}_{\mathcal{H}}$). Note that we can alternatively write the norm as

$$\begin{aligned} \|f+g\|_{\hat{\mathcal{X}}_+ + \hat{\mathcal{X}}_-} &= \inf\left\{ \sqrt{\|\tilde{f}\|_{\hat{\mathcal{X}}_+}^2 + \|\tilde{g}\|_{\hat{\mathcal{X}}_-}^2} \;\middle|\; \tilde{f} + \tilde{g} = f + g \right\} \\ &= \inf\left\{ \sqrt{\|f+z\|_{\hat{\mathcal{X}}_+}^2 + \|g-z\|_{\hat{\mathcal{X}}_-}^2} \;\middle|\; z \in \hat{\mathcal{X}}_+ \cap \hat{\mathcal{X}}_- \right\}. \end{aligned}$$

Moreover, the mapping

$$\Lambda\colon \begin{cases} \mathcal{X}_+ \times \mathcal{X}_- & \to \quad \mathcal{H}, \\ \qquad \begin{bmatrix} f \\ g \end{bmatrix} & \mapsto \quad \phi_{\mathcal{X}_+} f + \phi_{\mathcal{X}_-} g, \end{cases}$$

is continuous as composition of the continuous embeddings into $\mathcal{H}$ and the continuous addition in $\mathcal{H}$. Hence, $\ker \Lambda$ is closed in $\mathcal{X}_+ \times \mathcal{X}_-$ and the quotient space $\mathcal{X}_+ \times \mathcal{X}_- / _{\ker \Lambda}$ is a Hilbert space and is isometrically isomorphic to $\hat{\mathcal{X}}_+ + \hat{\mathcal{X}}_-$ with

$\|\cdot\|_{\hat{\mathcal{X}}_+ + \hat{\mathcal{X}}_-}$. The quotient mapping $\Lambda/_{\ker\Lambda}\colon \mathcal{X}_+ \times \mathcal{X}_-/_{\ker\Lambda} \to \mathcal{H}$ is injective and continuous, which implies that topology of $\|\cdot\|_{\hat{\mathcal{X}}_+ + \hat{\mathcal{X}}_-}$ is finer than the trace topology of $\mathcal{T}_{\mathcal{H}}$ on $\hat{\mathcal{X}}_+ + \hat{\mathcal{X}}_-$.

We can regard $\hat{\mathcal{Z}}_+ := \phi_{\mathcal{X}_0}(\mathcal{Z}_+) \subseteq \hat{\mathcal{X}}_0 \subseteq \mathcal{H}$ and endow this space with

$$\|z\|_{\hat{\mathcal{Z}}_+} := \sqrt{\|z\|^2_{\hat{\mathcal{X}}_+} + \|z\|^2_{\hat{\mathcal{X}}_-}} = \|\phi^{-1}_{\mathcal{X}_0} z\|_{\mathcal{Z}_+} \quad \text{for} \quad z \in \hat{\mathcal{Z}}_+.$$

Furthermore, we can define a new norm on $\mathcal{X}_0$ by $\|h\|_{\hat{\mathcal{Z}}_-} := \sup_{z\in\hat{\mathcal{Z}}_+\setminus\{0\}} \frac{|\langle h,z\rangle|_{\hat{\mathcal{X}}_0}}{\|z\|_{\hat{\mathcal{Z}}_+}}$. Note that every $h \in \mathcal{X}_0$ can be written as $h = f + g$, where $f \in D_+$ and $g \in D_-$, see Corollary 5.2. Hence, also every $h \in \hat{\mathcal{X}}_0$ can be written as $h = f + g$, where $f \in \phi_{\mathcal{X}_0}(D_+) = \phi_{\mathcal{X}_+}(D_+) \subseteq \hat{\mathcal{X}}_+ \cap \hat{\mathcal{X}}_0$ and $g \in \phi_{\mathcal{X}_0}(D_-) = \phi_{\mathcal{X}_-}(D_-) \subseteq \hat{\mathcal{X}}_0 \cap \hat{\mathcal{X}}_-$. We know by Corollary 5.9 for every $f + g \in \hat{\mathcal{X}}_0$ ($f \in \phi_{\mathcal{X}_0}(D_+)$, $g \in \phi_{\mathcal{X}_0}(D_-)$) that

$$\begin{aligned}
\|f+g\|_{\hat{\mathcal{Z}}_-} &= \|\phi^{-1}_{\mathcal{X}_0}(f+g)\|_{\mathcal{Z}_-} = \inf_{z\in\mathcal{Z}_+} \sqrt{\|\phi^{-1}_{\mathcal{X}_0}(f) + z\|^2_{\mathcal{X}_+} + \|\phi^{-1}_{\mathcal{X}_0}(g) - z\|^2_{\mathcal{X}_-}} \\
&= \inf_{z\in\hat{\mathcal{Z}}_+} \sqrt{\|\underbrace{\phi^{-1}_{\mathcal{X}_0}(f) + \phi^{-1}_{\mathcal{X}_0}(z)}_{=\phi^{-1}_{\mathcal{X}_+}(f+z)}\|^2_{\mathcal{X}_+} + \|\underbrace{\phi^{-1}_{\mathcal{X}_0}(g) - \phi^{-1}_{\mathcal{X}_0}(z)}_{=\phi^{-1}_{\mathcal{X}_-}(g-z)}\|^2_{\mathcal{X}_-}} \\
&= \inf_{z\in\hat{\mathcal{Z}}_+} \sqrt{\|f+z\|^2_{\hat{\mathcal{X}}_+} + \|g-z\|^2_{\hat{\mathcal{X}}_-}} \\
&\geq \inf_{z\in\mathcal{X}_+\cap\mathcal{X}_-} \sqrt{\|f+z\|^2_{\hat{\mathcal{X}}_+} + \|g-z\|^2_{\hat{\mathcal{X}}_-}} = \|f+g\|_{\hat{\mathcal{X}}_+ + \hat{\mathcal{X}}_-},
\end{aligned}$$

because $\hat{\mathcal{Z}}_+ \subseteq \hat{\mathcal{X}}_+ \cap \hat{\mathcal{X}}_-$. Hence, the completion of $\hat{\mathcal{X}}_0$ w.r.t. $\|\cdot\|_{\hat{\mathcal{Z}}_-}$ can also be continuously embedded into $\hat{\mathcal{X}}_+ + \hat{\mathcal{X}}_-$, because $\hat{\mathcal{X}}_+ + \hat{\mathcal{X}}_-$ is complete, and therefore also into $\mathcal{H}$. In particular the mapping ($\psi_{\mathcal{X}_0}$ does not do anything by assumption)

$$\phi_{\mathcal{X}_0} \circ \psi^{-1}_{\mathcal{X}_0}\colon \psi_{\mathcal{X}_0}(\mathcal{X}_0) \subseteq \mathcal{Z}_- \to \mathcal{H}$$

is continuous w.r.t. the $\|\cdot\|_{\mathcal{Z}_-}$ topology on $\psi_{\mathcal{X}_0}(\mathcal{X}_0)$ and $\mathcal{T}_{\mathcal{H}}$ on $\mathcal{H}$ and injective. By the density of $\mathcal{X}_0$ in $\mathcal{Z}_-$ we can continuously extend this mapping, denoted by

$$\phi_{\mathcal{Z}_-} := \overline{\phi_{\mathcal{X}_0} \circ \psi^{-1}_{\mathcal{X}_0}}\colon \mathcal{Z}_- \to \mathcal{H}.$$

By construction we already have $\phi_{\mathcal{Z}_-} \circ \psi_{\mathcal{X}_0} = \phi_{\mathcal{X}_0}$. Note that for $z \in \mathcal{Z}_+$ we have

$$z = \psi_{\mathcal{X}_+} z = \psi_{\mathcal{X}_0} z = \psi_{\mathcal{X}_-} z \quad \text{and} \quad \phi_{\mathcal{X}_+} z = \phi_{\mathcal{X}_0} z = \phi_{\mathcal{X}_-} z.$$

Now for $f \in \mathcal{X}_+$ there exists a sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathcal{Z}_+$ that converges to f w.r.t. $\|\cdot\|_{\mathcal{X}_+}$. Hence, the continuity of $\phi_{\mathcal{Z}_-}$, $\psi_{\mathcal{X}_+}$ and $\phi_{\mathcal{X}_+}$ gives

$$\begin{aligned}
\phi_{\mathcal{Z}_-} \circ \psi_{\mathcal{X}_+} f &= \lim_{n\to\infty} \phi_{\mathcal{Z}_-} \circ \psi_{\mathcal{X}_+} z_n = \lim_{n\to\infty} \phi_{\mathcal{Z}_-} \circ \psi_{\mathcal{X}_0} z_n \\
&= \lim_{n\to\infty} \phi_{\mathcal{X}_0} z_n = \lim_{n\to\infty} \phi_{\mathcal{X}_+} z_n = \phi_{\mathcal{X}_+} f.
\end{aligned}$$

Analogously, we can show $\phi_{\mathcal{Z}_-} \circ \psi_{\mathcal{X}_-} = \phi_{\mathcal{X}_-}$. ❑

Corollary 5.11. *Let $\mathcal{H}$ be a topological Hausdorff vector space such that we can structure preservingly embed the quasi Gelfand triple $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ into $\mathcal{H}$. Then $\mathcal{X}_+ \cap \mathcal{X}_-$ in $\mathcal{H}$ equals $D_+ \cap D_-$, i.e., $\phi_{\mathcal{X}_+}(\mathcal{X}_+) \cap \phi_{\mathcal{X}_-}(\mathcal{X}_-) = \phi_{\mathcal{X}_0}(D_+ \cap D_-)$.*

Proof. By Theorem 5.10 we can also embed $\mathcal{Z}_-$ into $\mathcal{H}$ such that

$$\begin{matrix}\mathcal{X}_+ \\ \mathcal{X}_-\end{matrix} \subseteq \mathcal{Z}_- \subseteq \mathcal{H}.$$

Hence, $\mathcal{X}_+ \cap \mathcal{X}_-$ in $\mathcal{H}$ is the same as $\mathcal{X}_+ \cap \mathcal{X}_-$ in $\mathcal{Z}_-$, which equals, by Theorem 5.8, $D_+ \cap D_- = \mathcal{Z}_+$. ❑

6. Gram operators

Every quasi Gelfand triple $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is fully determined (up to isomorphic identifications) by $\mathcal{X}_0$, $\operatorname{ran} \iota_+$ and $\|\cdot\|_{\mathcal{X}_+}$ on $\operatorname{ran} \iota_+$ (or $\operatorname{ran} \iota_-$ with $\|\cdot\|_{\mathcal{X}_-}$). However, in the Hilbert space case ($\mathcal{X}_+$ is a Hilbert space) we can even encode the entire information of a quasi Gelfand triple in a single (so called *Gram*) operator G on $\mathcal{X}_0$, that is self-adjoint, positive and injective. This means that $\langle Gf, g\rangle_{\mathcal{X}_0}$ defines a new inner product on $\mathcal{X}_0$, which gives rise to $\langle f, g\rangle_{\mathcal{X}_+}$. In particular, we will see that $D_+ = \operatorname{dom} G^{1/2}$ and $\langle G^{1/2} f, G^{1/2} g\rangle = \langle f, g\rangle_{\mathcal{X}_+}$.

Definition 6.1. Let $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ be a quasi Gelfand triple of Hilbert spaces. Then we define the *Gram operator* $G_+ \colon \operatorname{dom} G_+ \subseteq \mathcal{X}_0 \to \mathcal{X}_0$ of the quasi Gelfand triple by

$$G_+ \coloneqq (\iota_+^{-1})^{*_\mathrm{h}} \iota_+^{-1} = (\iota_+ \iota_+^{*_\mathrm{h}})^{-1},$$

where here the adjoint is taken w.r.t. the dual pairs $(\mathcal{X}_0, \mathcal{X}_0)$ and $(\mathcal{X}_+, \mathcal{X}_+)$, i.e., $(\iota_+^{-1})^{*_\mathrm{h}} = (\iota_+^{-1})^{*\mathcal{X}_+ \times \mathcal{X}_0}$ and $\iota_+^{*_\mathrm{h}} = \iota_+^{*\mathcal{X}_0 \times \mathcal{X}_+}$.

By Theorem B.2 G_+ is self-adjoint and positive (not necessarily strictly positive (coercive)). Moreover, by the functional calculus for unbounded self-adjoint operators on Hilbert spaces there exists a root $G_+^{1/2}$ of G_+, which is also self-adjoint and positive.

Clearly, we can do the same for ι_- and define $G_- \coloneqq (\iota_-^{-1})^{*_\mathrm{h}} \iota_-^{-1}$, where again here the adjoint is taken w.r.t. the dual pairs $(\mathcal{X}_0, \mathcal{X}_0)$ and $(\mathcal{X}_-, \mathcal{X}_-)$, i.e., $(\iota_-^{-1})^{*_\mathrm{h}} = (\iota_-^{-1})^{*\mathcal{X}_- \times \mathcal{X}_0}$. In fact we will see that $G_- = G_+^{-1}$.

Theorem 6.2. *Let $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ be a quasi Gelfand triple of Hilbert spaces and G_+ its Gram operator. Then $\operatorname{ran} \iota_+ = \operatorname{dom} G_+^{1/2}$ and*

$$\langle f, g\rangle_{\mathcal{X}_+} = \langle G_+^{1/2} f, G_+^{1/2} g\rangle_{\mathcal{X}_0} \quad \text{for all} \quad f, g \in \operatorname{dom} G_+^{1/2}.$$

In particular, $\|f\|_{\mathcal{X}_+} = \|G_+^{1/2} f\|_{\mathcal{X}_0}$.

This theorem is basically a consequence of the *second representation theorem* for closed sesquilinear forms, see [9, Ch. VI, Thm. 2.23]. However, for convenience we present a proof.

Proof. Note that $\operatorname{dom} G_+ = \operatorname{dom}(\iota_+^{-1})^{*_\mathrm{h}} \iota_+^{-1}$ is a core of ι_+^{-1} and of $G_+^{1/2}$.

For $f, g \in \operatorname{dom} G_+ \subseteq \operatorname{dom} G_+^{1/2}$ we have

$$\big\langle \iota_+^{-1} f, \iota_+^{-1} g\big\rangle_{\mathcal{X}_+} = \big\langle (\iota_+^{-1})^{*_\mathrm{h}} \iota_+^{-1} f, g\big\rangle_{\mathcal{X}_0} = \langle G_+ f, g\rangle_{\mathcal{X}_0} = \langle G_+^{1/2} f, G_+^{1/2} g\rangle_{\mathcal{X}_0} \tag{6}$$

and in particular we have $\|\iota_+^{-1} f\|_{\mathcal{X}_+} = \|G_+^{1/2} f\|_{\mathcal{X}_0}$ for all $f \in \operatorname{dom} G_+$.

Since $\operatorname{dom} G^{1/2}$ is a core, for every $f \in \operatorname{dom} \iota_+^{-1}$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $\operatorname{dom} G_+$ that converges to f w.r.t. the graph norm of ι_+^{-1}. Hence, we have

$$\|G_+^{1/2}(f_n - f_m)\|_{\mathcal{X}_0} = \|\iota_+^{-1}(f_n - f_m)\|_{\mathcal{X}_+} \to 0$$

for $m, n \to \infty$, which implies that $(G_+^{1/2} f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and therefore convergent. By the closedness of $G_+^{1/2}$ the limit is $G_+^{1/2} f$. In particular, $\operatorname{ran} \iota_+ =$

$\operatorname{dom} \iota_+^{-1} \subseteq \operatorname{dom} G_+^{1/2}$. The same argument with $G_+^{1/2}$ and ι_+^{-1} swapped gives equality, i.e., $\operatorname{ran} \iota_+ = \operatorname{dom} \iota_+^{-1} = \operatorname{dom} G_+^{1/2}$.

Finally, we can extend (6) to $\operatorname{dom} G_+^{1/2}$ by continuity. ❑

Proposition 6.3. *Let $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ be a quasi Gelfand triple of Hilbert spaces. Then*

$$G_- = G_+^{-1}.$$

Proof. Let $\Psi\colon \mathcal{X}_- \to \mathcal{X}_+$ denote the duality mapping between $\mathcal{X}_-$ and $\mathcal{X}_+$. Recall $G_- = (\iota_-^{-1})^{*_\mathrm{h}} \iota_-^{-1} = (\iota_- \iota_-^{*_\mathrm{h}})^{-1}$,

$$\iota_+^{*_\mathrm{h}} = \Psi \iota_+^* = \Psi \iota_-^{-1} \quad \text{and} \quad \iota_-^{*_\mathrm{h}} = \Psi^{-1} \iota_-^* = \Psi^{-1} \iota_+^{-1}.$$

Hence, we have

$$G_-^{-1} = \iota_- \iota_-^{*_\mathrm{h}} = \iota_- \Psi^{-1} \iota_+^{-1} = (\Psi \iota_-^{-1})^{-1} \iota_+^{-1} = (\iota_+^{*_\mathrm{h}})^{-1} \iota_+^{-1} = (\iota_+^{-1})^{*_\mathrm{h}} \iota_+^{-1} = G_+.$$ ❑

Corollary 6.4. *Let $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ be a quasi Gelfand triple of Hilbert spaces. Then*

$$\operatorname{ran} \iota_- = \operatorname{dom} G_-^{1/2} = \operatorname{dom} G_+^{-1/2} = \operatorname{ran} G_+^{1/2}.$$

So far we have shown that there is a self-adjoint positive and injective operator with dense range for every quasi Gelfand triple. Now the next theorem will show that also the reverse is true. That is, every self-adjoint positive and injective operator G with dense range establishes a quasi Gelfand triple whose Gram operator is G.

Theorem 6.5. *Let $\mathcal{X}_0$ be a Hilbert space and G a self-adjoint positive and injective operator on $\mathcal{X}_0$ with dense range. Then there exists a quasi Gelfand triple whose Gram operator is G. In particular, if we denote the corresponding quasi Gelfand triple by $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ we have*

$$\operatorname{ran} \iota_+ = \operatorname{dom} G^{1/2} \quad \text{and} \quad \operatorname{ran} \iota_- = \operatorname{ran} G^{1/2}.$$

Moreover, G coincides with the Gram operator G_+ of $(\mathcal{X}_+, \mathcal{X}_0\, \mathcal{X}_-)$, i.e., $G = G_+$.

Note that dense range and injectivity are equivalent for a self-adjoint operator. Moreover, the density of the range (or the injectivity of the operator) is not really a necessity as we can always split

$$\mathcal{X}_0 = \ker G \oplus \overline{\operatorname{ran} G}.$$

Hence, we just replace $\mathcal{X}_0$ with $\overline{\operatorname{ran} G}$ and G with $G|_{\overline{\operatorname{ran} G}}$.

Proof. We define $\langle f, g\rangle_{\mathcal{X}_+} := \langle G^{1/2} f, G^{1/2} g\rangle_{\mathcal{X}_0}$ and the corresponding norm $\|f\|_{\mathcal{X}_+} = \|G^{1/2} f\|_{\mathcal{X}_0}$ for $f, g \in \operatorname{dom} G^{1/2}$. Since $G^{1/2}$ is positive $\langle\cdot,\cdot\rangle_{\mathcal{X}_+}$ is really an inner product and $\|\cdot\|_{\mathcal{X}_+}$ a norm. Hence, $\operatorname{dom} G^{1/2}$ with $\langle\cdot,\cdot\rangle_{\mathcal{X}_+}$ is a pre-Hilbert space and its completion $\mathcal{X}_+$ is a Hilbert space. We define

$$\iota_+ \colon \begin{cases} \operatorname{dom} G^{1/2} \subseteq \mathcal{X}_+ & \to & \mathcal{X}_0, \\ f & \mapsto & f. \end{cases}$$

Let $\left(\begin{bmatrix} f_n \\ f_n \end{bmatrix}\right)_{n\in\mathbb{N}}$ be a sequence in ι_+ that converges to $\begin{bmatrix} g \\ f \end{bmatrix} \in \mathcal{X}_+ \times \mathcal{X}_0$. Then $\left(\begin{bmatrix} f_n \\ G^{1/2} f_n \end{bmatrix}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}_0 \times \mathcal{X}_0$, and therefore convergent. The closedness of $G^{1/2}$ implies $f \in \operatorname{dom} G^{1/2} = D_+$ and $\begin{bmatrix} f_n \\ G^{1/2} f_n \end{bmatrix} \to \begin{bmatrix} f \\ G^{1/2} f \end{bmatrix}$. This leads to $\|f_n - f\|_{\mathcal{X}_+} = \|G^{1/2}(f_n - f)\|_{\mathcal{X}_0} \to 0$ and consequently $f = g$. Now we can apply Theorem 4.8 and see that there is a space $\mathcal{X}_-$ such that $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ forms a quasi Gelfand triple.

Now we have for $f,g\in\operatorname{dom}G^{1/2}=\operatorname{ran}\iota_+=\operatorname{dom}G_+^{1/2}$

$$\langle G^{1/2}f,G^{1/2}g\rangle_{\mathcal{X}_0}=\langle f,g\rangle_{\mathcal{X}_+}=\langle G_+^{1/2}f,G_+^{1/2}g\rangle_{\mathcal{X}_0}.$$

Note that $\operatorname{dom}G\subseteq\operatorname{dom}G^{1/2}$ and therefore for $f\in\operatorname{dom}G$ we have

$$\langle Gf,g\rangle_{\mathcal{X}_0}=\langle G_+^{1/2}f,G_+^{1/2}g\rangle_{\mathcal{X}_0},$$

which implies $G_+^{1/2}f\in\operatorname{dom}G_+^{1/2}$ and $G_+^{1/2}G_+^{1/2}f=Gf$. Hence, $G\subseteq G_+$. Since both G and G_+ are self-adjoint they have to coincide.

By Proposition 6.3 we have $G_-=G_+^{-1}=G^{-1}$ and therefore, by Theorem 6.2 for G_-,

$$\operatorname{ran}\iota_-=\operatorname{dom}G_-^{1/2}=\operatorname{ran}G_-^{-1/2}=\operatorname{ran}G^{1/2}.$$ ❑

There is a bijection between the set of quasi Gelfand triples with pivot space $\mathcal{X}_0$ and all self-adjoint positive and injective operators with dense range on $\mathcal{X}_0$, see Figure 5.

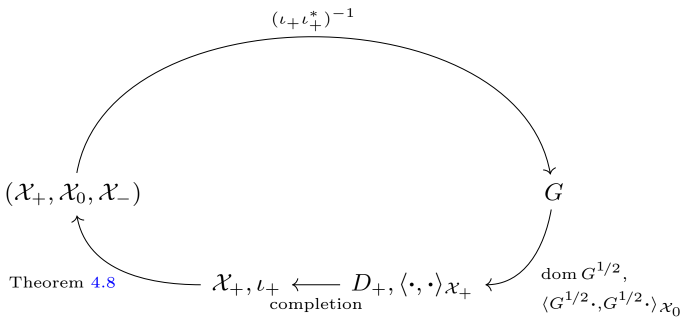


FIGURE 5. Illustration of Theorem 6.5

Since all infinite dimensional separable Hilbert spaces are isomorphic, it is clear that there exists a dual pairing $\langle\cdot,\cdot\rangle_{\mathcal{X}_+,\mathcal{X}_0}$ such that also $(\mathcal{X}_+,\mathcal{X}_0)$ is a complete dual pair. However, we can even explicitly write this dual pairing by

$$\langle f,g\rangle_{\mathcal{X}_+,\mathcal{X}_0}=\left\langle\overline{G_+^{1/2}\iota_+}f,g\right\rangle_{\mathcal{X}_0}=\left\langle f,\overline{G_+^{1/2}\iota_+}^{-1}g\right\rangle_{\mathcal{X}_+},$$

where $\overline{G_+^{1/2}\iota_+}$ denotes the continuous extension of the isometric mapping $G_+^{1/2}\iota_+\colon\operatorname{dom}\iota_+\subseteq\mathcal{X}_+\to\mathcal{X}_0$.

6.1. Decomposition into two "ordinary" Gelfand triples

In this section we will see that every quasi Gelfand triple of Hilbert spaces can be decomposed into two "ordinary" Gelfand triples. This means for a quasi Gelfand triple $(\mathcal{X}_+,\mathcal{X}_0,\mathcal{X}_-)$ there exist "ordinary" Gelfand triples $\mathcal{X}_+^1\subseteq\mathcal{X}_0^1\subseteq\mathcal{X}_-^1$ and $\mathcal{X}_+^2\subseteq\mathcal{X}_0^2\subseteq\mathcal{X}_-^2$ such that

$$\mathcal{X}_+=\mathcal{X}_+^1\oplus\mathcal{X}_-^2,\quad \mathcal{X}_0=\mathcal{X}_0^1\oplus\mathcal{X}_0^2\quad\text{and}\quad\mathcal{X}_-=\mathcal{X}_-^1\oplus\mathcal{X}_+^2.$$

Theorem 6.6. *Let $(\mathcal{X}_+,\mathcal{X}_0,\mathcal{X}_-)$ be a quasi Gelfand triple of Hilbert spaces. Then there exist two "ordinary" Gelfand triples $\mathcal{X}_+^1\subseteq\mathcal{X}_0^1\subseteq\mathcal{X}_-^1$ and $\mathcal{X}_+^2\subseteq\mathcal{X}_0^2\subseteq\mathcal{X}_-^2$ such that*

$$\mathcal{X}_+=\mathcal{X}_+^1\oplus\mathcal{X}_-^2,\quad \mathcal{X}_0=\mathcal{X}_0^1\oplus\mathcal{X}_0^2\quad\textit{and}\quad\mathcal{X}_-=\mathcal{X}_-^1\oplus\mathcal{X}_+^2.$$

This means that every quasi Gelfand triple (of Hilbert spaces) is the result of two "ordinary" Gelfand triple that are cross-wise composed.

Proof. We will show the proof in several steps:

1. Step: *Decomposition of* $\mathcal{X}_0$. Let G_+ be the Gram operator of the quasi Gelfand triple and $G_+^{1/2}$ its positive square root. Then there exists a spectral measure E for $G_+^{1/2}$ such that $G_+^{1/2} = \int_{\mathbb{R}_+} \lambda \,\mathrm{d}E(\lambda)$. We can decompose $\mathcal{X}_0$ into

$$\mathcal{X}_0 = \underbrace{\operatorname{ran} E((1,\infty))}_{=:\mathcal{X}_0^1} \oplus \underbrace{\operatorname{ran} E((0,1])}_{=:\mathcal{X}_0^2}.$$

By spectral theory $\mathcal{X}_0^2 = \operatorname{ran} E((0,1]) \subseteq \operatorname{dom} G_+^{1/2} = \operatorname{ran} \iota_+ = D_+$, as $(0,1]$ is a bounded set. We can write every $f \in D_+$ as

$$f = E((1,\infty))f + E((0,1])f$$

and since $E((0,1])f \in D_+$, we conclude that also $E((1,\infty))f \in D_+$. For an arbitrary $f \in \mathcal{X}_0^1 \subseteq \mathcal{X}_0$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in D_+ such that $f_n \to f$ w.r.t. $\|\cdot\|_{\mathcal{X}_0}$. Since also $(E((1,\infty))f_n)_{n\in\mathbb{N}}$ converges to $E((1,\infty))f = f$ by continuity, and $E((1,\infty))f \in D_+$, we conclude that $\mathcal{X}_0^1 \cap D_+$ is dense in $\mathcal{X}_0^1$ (w.r.t. $\|\cdot\|_{\mathcal{X}_0}$). On the other hand, $\mathcal{X}_0^2 \subseteq D_+$.

2. Step: *Decomposition of* $\mathcal{X}_+$. For $f \in D_+$ we have

$$\begin{aligned}\|E((0,1])f\|_{\mathcal{X}_+}^2 &= \|G_+^{1/2}E((0,1])f\|_{\mathcal{X}_0}^2 \\ &= \int_{(0,1]} |\lambda|^2 \,\mathrm{d}E_{f,f} \leq \int_{(0,\infty)} |\lambda|^2 \,\mathrm{d}E_{f,f} = \|f\|_{\mathcal{X}_+}^2,\end{aligned}$$

and

$$\begin{aligned}\|E((1,\infty))f\|_{\mathcal{X}_+}^2 &= \|G_+^{1/2}E((1,\infty))f\|_{\mathcal{X}_0}^2 \\ &= \int_{(1,\infty)} |\lambda|^2 \,\mathrm{d}E_{f,f} \leq \int_{(0,\infty)} |\lambda|^2 \,\mathrm{d}E_{f,f} = \|f\|_{\mathcal{X}_+}^2.\end{aligned}$$

Hence, the spectral projections $E((0,1])$ and $E((1,\infty))$ are also continuous on D_+ with respect to $\|\cdot\|_{\mathcal{X}_+}$ and we can extend these projections continuously on $\mathcal{X}_+$. Note that for $f \in D_+$ we have $G^{1/2}E(\Delta)f = E(\Delta)G^{1/2}f$ for all Δ in the Borel sets of $\mathbb{R}$. Hence, we have for $f, g \in D_+$

$$\begin{aligned}\langle E((0,1])f, E((1,\infty))g\rangle_{\mathcal{X}_+} &= \langle G_+^{1/2}E((0,1])f, G_+^{1/2}E((1,\infty))g\rangle_{\mathcal{X}_0} \\ &= \langle G_+\underbrace{E((1,\infty))E((0,1])}_{=0}f, g\rangle_{\mathcal{X}_0} = 0,\end{aligned}$$

which implies that the extensions of $E((0,1])\big|_{D_+}$ and $E((1,\infty))\big|_{D_+}$ are orthogonal projections on $\mathcal{X}_+$. Moreover, for $f \in \mathcal{X}_+$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in D_+ that converges to f w.r.t. $\|\cdot\|_{\mathcal{X}_+}$. By the continuity of projections we conclude that $(E((0,1])f_n)_{n\in\mathbb{N}}$ and $(E((1,\infty))f_n)_{n\in\mathbb{N}}$ converge and therefore

$$\begin{aligned}f = \lim_{n\to\infty} f_n &= \lim_{n\to\infty} E((0,1])f_n + E((1,\infty))f_n \\ &= \lim_{n\to\infty} E((0,1])f_n + \lim_{n\to\infty} E((1,\infty))f_n.\end{aligned}$$

This leads to: the extensions of these projections are also complementary. We denote these extensions by $E((0,1])_+$ and $E((1,\infty))_+$ and we have

$$\mathcal{X}_+ = \underbrace{\operatorname{ran} E((1,\infty))_+}_{=:\mathcal{X}_+^1} \oplus \underbrace{\operatorname{ran} E((0,1])_+}_{=:\mathcal{X}_-^2}.$$

3. Step: *Relationship between the decompositions of* $\mathcal{X}_0$ *and* $\mathcal{X}_+$*.* Note that we have $E((1,\infty))_+ D_+ = E((1,\infty)) D_+ = \mathcal{X}_0^1 \cap D_+$. Furthermore, for $f \in \mathcal{X}_0^1 \cap D_+$ we have

$$\begin{aligned} \|f\|_{\mathcal{X}_+}^2 &= \|E((1,\infty))f\|_{\mathcal{X}_+}^2 = \|G_+^{1/2} E((1,\infty)) f\|_{\mathcal{X}_0}^2 \\ &= \int_{(1,\infty)} |\lambda|^2 \,\mathrm{d}E_{f,f} \geq \inf_{\lambda \in (1,\infty)} |\lambda|^2 \|f\|_{\mathcal{X}_0}^2 \geq \|f\|_{\mathcal{X}_0}^2. \end{aligned} \quad (7)$$

Now for $f \in \mathcal{X}_+^1$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in D_+ that converges to f w.r.t. $\|\cdot\|_{\mathcal{X}_+}$ and therefore also $(\tilde{f}_n)_{n\in\mathbb{N}} = (E((1,\infty))_+ f_n)_{n\in\mathbb{N}}$ converges to f w.r.t. $\|\cdot\|_{\mathcal{X}_+}$. By (7) we have

$$\|\tilde{f}_n - \tilde{f}_m\|_{\mathcal{X}_0} \leq \|\tilde{f}_n - \tilde{f}_m\|_{\mathcal{X}_+} \to 0,$$

which implies that $(\tilde{f}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}_0^1$ (w.r.t. $\|\cdot\|_{\mathcal{X}_0}$). By the closedness of ι_+ the limit of this sequence (w.r.t. $\|\cdot\|_{\mathcal{X}_0}$) has to coincide with f. Hence, $\mathcal{X}_+^1 = \mathcal{X}_0^1 \cap D_+$ and the restricted embedding $\iota_+\big|_{\mathcal{X}_+^1} : \mathcal{X}_+^1 \to \mathcal{X}_0^1$ is continuous.

On the other hand, since $\mathcal{X}_0^2 \subseteq D_+$ we automatically have $\mathcal{X}_0^2 \subseteq \mathcal{X}_-^2$, by construction. Furthermore, for $f \in \mathcal{X}_0^2$ we have

$$\begin{aligned} \|f\|_{\mathcal{X}_+}^2 &= \|E((0,1])f\|_{\mathcal{X}_+}^2 = \|G_+^{1/2} E((0,1]) f\|_{\mathcal{X}_0}^2 \\ &= \int_{(0,1]} |\lambda|^2 \,\mathrm{d}E_{f,f} \leq \sup_{\lambda \in (0,1]} |\lambda|^2 \|f\|_{\mathcal{X}_0}^2 \leq \|f\|_{\mathcal{X}_0}^2. \end{aligned} \quad (8)$$

This implies that the inverse embedding ι_+^{-1} restricted to $\mathcal{X}_0^2$ is continuous, i.e., $\iota_+^{-1}\big|_{\mathcal{X}_0^2} : \mathcal{X}_0^2 \to \mathcal{X}_-^2$ is continuous. Hence, we have

$$\mathcal{X}_0^2 \subseteq \mathcal{X}_-^2 \quad \text{and} \quad \mathcal{X}_+^1 \subseteq \mathcal{X}_0^1$$

densely with continuous embeddings.

4. Step: *Decomposition of* $\mathcal{X}_-$*.* Note that for $g \in D_-$ we have

$$\|g\|_{\mathcal{X}_-} = \|G_-^{1/2} g\|_{\mathcal{X}_0} = \|G_+^{-1/2} g\|_{\mathcal{X}_0}$$

and additionally by the rules for the spectral calculus we have

$$G_-^{1/2} = G_+^{-1/2} = \int_{(0,\infty)} \frac{1}{\lambda} \,\mathrm{d}E.$$

Hence, exactly the same construction as in the second step (replace $\mathcal{X}_+$ by $\mathcal{X}_-$, D_+ by D_-, G_+ by G_- and $|\lambda|$ by $|\frac{1}{\lambda}|$) gives the decomposition

$$\mathcal{X}_- = \underbrace{\operatorname{ran} E((1,\infty))_-}_{=:\mathcal{X}_-^1} \oplus \underbrace{\operatorname{ran} E((0,1])_-}_{=:\mathcal{X}_+^2}.$$

5. Step: *Relation ship between the decompositions of $\mathcal{X}_0$ and $\mathcal{X}_-$.* Again repeating the arguments of the third step. In particular, for $g \in D_-$ we have

$$\begin{aligned}\|E((0,1])g\|_{\mathcal{X}_-}^2 &= \|G_-^{1/2}E((0,1])g\|_{\mathcal{X}_0}^2 \\ &= \int_{(0,1]} \left|\frac{1}{\lambda}\right|^2 \mathrm{d}E_{g,g} \geq \inf_{\lambda\in(0,1]} \left|\frac{1}{\lambda}\right|^2 \|g\|_{\mathcal{X}_0}^2 = \|g\|_{\mathcal{X}_0}^2\end{aligned}$$

and

$$\begin{aligned}\|E((1,\infty))g\|_{\mathcal{X}_-}^2 &= \|G_-^{1/2}E((1,\infty))g\|_{\mathcal{X}_0}^2 \\ &= \int_{(1,\infty)} \left|\frac{1}{\lambda}\right|^2 \mathrm{d}E_{g,g} \leq \inf_{\lambda\in(1,\infty)} \left|\frac{1}{\lambda}\right|^2 \|g\|_{\mathcal{X}_0}^2 = \|g\|_{\mathcal{X}_0}^2.\end{aligned}$$

This implies $\iota_-\big|_{\mathcal{X}_+^2} : \mathcal{X}_+^2 \to \mathcal{X}_0^2$ and $\iota_-^{-1}\big|_{\mathcal{X}_0^1} : \mathcal{X}_0^1 \to \mathcal{X}_-^1$ are continuous. In particular, we have

$$\mathcal{X}_+^2 \subseteq \mathcal{X}_0^2 \quad \text{and} \quad \mathcal{X}_0^1 \subseteq \mathcal{X}_-^1$$

densely with continuous embeddings.

6. Step: *Dualities.* By Hahn-Banach we can identify $(\mathcal{X}_-^2)'$ with $\mathcal{X}_+'\big|_{\mathcal{X}_-^2}$. Moreover, for $f \in \mathcal{X}_+$ and $g \in \mathcal{X}_-$ there exist sequences $(f_n)_{n\in\mathbb{N}}$ in D_+ and $(g_n)_{n\in\mathbb{N}}$ in D_- such that $f_n \to f$ w.r.t. $\|\cdot\|_{\mathcal{X}_+}$ and $g_n \to g$ w.r.t. $\|\cdot\|_{\mathcal{X}_-}$. Hence,

$$\begin{aligned}\langle E((0,1])_+ f, E((1,\infty))_- g\rangle_{\mathcal{X}_+,\mathcal{X}_-} &= \lim_{n\to\infty} \langle E((0,1])_+ f_n, E((1,\infty))_- g_n\rangle_{\mathcal{X}_+,\mathcal{X}_-} \\ &= \lim_{n\to\infty} \langle E((0,1]) f_n, E((1,\infty)) g_n\rangle_{\mathcal{X}_0} = 0.\end{aligned}$$

Clearly, we also have $\langle E((1,\infty))_+ f, E((0,1])_- g\rangle_{\mathcal{X}_+,\mathcal{X}_-} = 0$. For $\phi \in (\mathcal{X}_-^2)'$ there exists a $g \in \mathcal{X}_-$ such that

$$\phi(f) = \langle g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} = \langle E((0,1])_- g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} + \underbrace{\langle E((1,\infty))_- g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+}}_{=0} \ \forall f \in \mathcal{X}_-^2.$$

Moreover,

$$\begin{aligned}\|\phi\|_{(\mathcal{X}_-^2)'} &= \sup_{f\in\mathcal{X}_-^2\setminus\{0\}} \frac{|\phi(f)|}{\|f\|_{\mathcal{X}_+}} = \sup_{f\in\mathcal{X}_-^2\setminus\{0\}} \frac{|\langle E((0,1])_- g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+}|}{\|f\|_{\mathcal{X}_+}} \\ &= \sup_{f\in\mathcal{X}_+\setminus\{0\}} \frac{|\langle E((0,1])_- g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+}|}{\|f\|_{\mathcal{X}_+}} = \|E((0,1])_- g\|_{\mathcal{X}_-}\end{aligned}$$

On the other hand, if $\langle E((0,1])_- g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} = 0$ for all $f \in \mathcal{X}_-^2$, then we automatically have $\langle E((0,1])_- g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} = 0$ for all $f \in \mathcal{X}_+$ and therefore $E((0,1])_- g = 0$. In conclusion $(\mathcal{X}_-^2, \mathcal{X}_+^2)$ is a complete dual pair and $(\mathcal{X}_-^2, \mathcal{X}_0^2, \mathcal{X}_+^2)$ is a quasi Gelfand triple with the embeddings $\iota_+\big|_{\mathcal{X}_-^2}$ and $\iota_-\big|_{\mathcal{X}_+^2}$. Moreover, since $\iota_-\big|_{\mathcal{X}_+^2}$ is continuous, it is even an "ordinary" Gelfand triple $(\mathcal{X}_+^2 \subseteq \mathcal{X}_0^2 \subseteq \mathcal{X}_-^2)$.

We can show completely analogously that also $(\mathcal{X}_+^1, \mathcal{X}_0^1, \mathcal{X}_-^1)$ is an "ordinary" Gelfand triple $(\mathcal{X}_+^1 \subseteq \mathcal{X}_0^1 \subseteq \mathcal{X}_-^1)$. ❑

Note that this decomposition is not unique as we could have split the space $\mathcal{X}_0$ by any two subspaces $\operatorname{ran} E(\Delta)$ and $\operatorname{ran} E(\Delta^{\complement})$, where $(0,\epsilon) \subseteq \Delta \subseteq \mathbb{R}_+$ is a bounded non-empty Borel set for any $\epsilon > 0$.

Finally, we end with two conjectures

Conjecture 6.7 (weak). Every pre-quasi Gelfand triple of Hilbert spaces is a quasi Gelfand triple.

Conjecture 6.8 (strong). Every pre-quasi Gelfand triple is a quasi Gelfand triple.

At least the weak conjecture seems to be true, but all attempts failed so far. In fact Theorem 5.10 and Theorem 6.6 are the result of failed attempts to prove the weak conjecture. The strong conjecture seems much more difficult, as a lot of Hilbert space theory is unavailable.

A positive answer to (at least) the weak conjecture would automatically answer the question whether the weak and strong definition of boundary trace operators for differential operators coincide.

Conclusion

We have introduced a generalization of Gelfand triple that does not need continuous embeddings. This was done by replacing the continuity of the embeddings by closedness. We showed that $D_+ \cap D_-$, the set that is in the intersection of the quasi Gelfand triple, is dense in the pivot space $\mathcal{X}_0$.

If we regard quasi Gelfand triples of Hilbert spaces, then we can show that $D_+ \cap D_-$ is also dense in $\mathcal{X}_+$ and $\mathcal{X}_-$ w.r.t. their norms. Furthermore, we have shown that there exists a smallest space were we can embed the entire quasi Gelfand triple structure preservingly.

Finally, we have shown that every quasi Gelfand triple is associated to a Gram operator and the other way round. This led us to a decomposition of the quasi Gelfand triple into two "ordinary" Gelfand triples.

We ended with the weak and strong version of the conjecture that every pre-quasi Gelfand triple is in fact already a quasi Gelfand triple.

One application that we did not cover, that is still worth mentioning: Quasi Gelfand triples can be used to properly define boundary spaces and characterizing suitable boundary conditions for partial differential equations that lead to existence and uniqueness of solutions, see [13].

Declarations

Conflict of interest. The authors have no relevant financial or non-financial interests to disclose.

Data availability. This work did not involve any underlying data

Appendix A. Comparison to similar concepts

In this section we want to introduce the notions *triplets of spaces* and *triplets of closely embedded Hilbert spaces*[1] and compare them to quasi Gelfand triples. We will show that all these notions coincide, i.e.,

$$\begin{aligned}&(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-) \text{ is a quasi Gelfand triple}\\ &\qquad \Leftrightarrow \quad (\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-) \text{ is a triplet of spaces}\\ &\qquad \Leftrightarrow \quad (\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-) \text{ is a triplet of closely embedded Hilbert spaces.}\end{aligned}$$

[1]We stick to the original naming of the authors, i.e., "triplet" instead of "triple"

In [6] they investigated the equivalence between *triplets of spaces* and *triplets of closely embedded Hilbert spaces* and gave conditions for their equivalence. Suprisingly, they did not realize that no conditions are needed.

First we state the original definition of *triplets of spaces*, which includes some implicit assumptions on how to understand intersections of different Hilbert spaces that are not embedded in a common space.

Definition A.1 (Triplets of spaces (original)). Let $\mathcal{X}_+$, $\mathcal{X}_0$, $\mathcal{X}_-$ be three Hilbert spaces. We say $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a *triplet of spaces*, if the following assertions hold.

(a) $D \coloneqq \mathcal{X}_+ \cap \mathcal{X}_0 \cap \mathcal{X}_-$ is dense in each of these spaces.

(b) The sesquilinear form $B(g,f) \coloneqq \langle g, f\rangle_{\mathcal{X}_0}$ admits the estimate

$$|B(g,f)| \leq \|g\|_{\mathcal{X}_-} \|f\|_{\mathcal{X}_+} \quad \text{for all} \quad g,f \in D.$$

We denote the continuous extension of B to $\mathcal{X}_- \times \mathcal{X}_+$ still by B.

(c) For every $h \in \mathcal{X}_+$ there exists a unique $g_h \in \mathcal{X}_-$ such that $\langle h, f\rangle_{\mathcal{X}_+} = B(g_h, f)_{\mathcal{X}_0}$ for all $f \in \mathcal{X}_+$.

However, in order to avoid misinterpretation of the meaning of the intersections in the previous definition, we introduce the following clarification.

Definition A.2 (Triplets of spaces (clarified)). Let $\mathcal{X}_+$, $\mathcal{X}_0$, $\mathcal{X}_-$ be three Hilbert spaces. We say $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a *triplet of spaces*, if there exist mappings

$$k_+ \colon \operatorname{dom} k_+ \subseteq \mathcal{X}_+ \to \mathcal{X}_0 \quad \text{and} \quad k_- \colon \operatorname{dom} k_- \subseteq \mathcal{X}_- \to \mathcal{X}_0$$

linear and injective such that the following assertions hold.

(tos1) $\mathcal{X}_+ \cap \mathcal{X}_0 \cap \mathcal{X}_-$ is dense in each of these spaces, i.e.,

$$\begin{aligned} D \coloneqq \operatorname{ran} k_+ \cap \operatorname{ran} k_- \quad &\text{is dense in} \quad \mathcal{X}_0, \\ k_+^{-1}(D) = k_+^{-1}\big(\operatorname{ran} k_+ \cap \operatorname{ran} k_-\big) \quad &\text{is dense in} \quad \mathcal{X}_+, \\ k_-^{-1}(D) = k_-^{-1}\big(\operatorname{ran} k_+ \cap \operatorname{ran} k_-\big) \quad &\text{is dense in} \quad \mathcal{X}_-. \end{aligned}$$

(tos2) The sesquilinear form $B(g,f) \coloneqq \langle k_- g, k_+ f\rangle_{\mathcal{X}_0}$ admits the estimate

$$|B(g,f)| \leq \|g\|_{\mathcal{X}_-} \|f\|_{\mathcal{X}_+} \quad \text{for all} \quad g \in k_-^{-1}(D), f \in k_+^{-1}(D),$$

where we denote the continuous extension of B to $\mathcal{X}_- \times \mathcal{X}_+$ still by B.

(tos3) For every $h \in \mathcal{X}_+$ there exists a unique $g \in \mathcal{X}_-$ such that $\langle h, f\rangle_{\mathcal{X}_+} = B(g,f)$ for all $f \in \mathcal{X}_+$.

For one direction of the equivalence between quasi Gelfand triples and triplets of spaces we are already prepared.

Lemma A.3. *Let $\mathcal{X}_+$, $\mathcal{X}_0$, $\mathcal{X}_-$ be Hilbert spaces. If $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is quasi Gelfand triple, then $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a triplet of spaces.*

Proof. Note that since every quasi Gelfand triple is structure preservingly embedded in $\mathcal{Z}_-$ (Theorem 5.5), we can omit the embedding operators. However, if we would like to be more precise we would just define $k_+ = \iota_+$ and $k_- = \iota_-$.

The set $D = \mathcal{X}_+ \cap \mathcal{X}_0 \cap \mathcal{X}_- = D_+ \cap D_-$ is dense in $\mathcal{X}_+$ and $\mathcal{X}_-$ by Corollary 5.1 and dense in $\mathcal{X}_0$ by Proposition 4.12. Furthermore,

$$|B(g,f)| = |\langle g, f\rangle_{\mathcal{X}_0}| = |\langle g, f\rangle_{\mathcal{X}_-, \mathcal{X}_+}| \leq \|g\|_{\mathcal{X}_-} \|f\|_{\mathcal{X}_+}. \qquad ❑$$

The reverse direction needs more attention. Especially, because the notion of triplets of spaces leaves room for ambiguity. Hence, we first want to highlight what we mean by this ambiguity.

Example A.4. Let $w\colon (0,1) \to (0,\infty)$ be a measurable and (essentially) unbounded function such that also $\frac{1}{w}$ is (essentially) unbounded, e.g., $w(x) = \frac{1-x}{x}$ and $v\colon (0,1) \to (0,1)$ be a measurable and (essentially) bounded function such that also $\frac{1}{v}$ is (essentially) bounded, e.g., $v(x) = \frac{1}{2}$. Then we define weighted L^2 spaces

$$\mathcal{X}_+ = \mathrm{L}^2\big((0,1), w \,\mathrm{d}\lambda\big), \quad \mathcal{X}_0 = \mathrm{L}^2\big((0,1), v \,\mathrm{d}\lambda\big) \quad \text{and} \quad \mathcal{X}_- = \mathrm{L}^2\big((0,1), \tfrac{1}{w} \,\mathrm{d}\lambda\big),$$

where λ denotes the Lebesgue measure. It is straightforward to show that $(\mathcal{X}_+, \mathcal{X}_-)$ is a (complete) dual pair with the dual pairing $\langle g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} = \int_{(0,1)} g\overline{f} \,\mathrm{d}\lambda$. Note that the intersection of all these spaces contains $\mathrm{C}_\mathrm{c}^\infty\big((0,1)\big)$ which is dense in all of these spaces. Moreover,

$$B(g,f) = \langle g, f\rangle_{\mathcal{X}_0} = \int_{(0,1)} \tfrac{1}{w} g w \overline{f} v \,\mathrm{d}\lambda \leq \underbrace{\|v\|_\infty}_{\leq 1} \|g\|_{\mathcal{X}_-} \|f\|_{\mathcal{X}_+}.$$

Hence, $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a triplet of spaces. However, this is true for every v and therefore the space $\mathcal{X}_0$—or more precisely its inner product—is not uniquely determined. In order to obtain a quasi Gelfand triple we either have to choose a different (but equivalent) inner product for $\mathcal{X}_0$ or for $\mathcal{X}_-$.

The previous example shows that in general we have to expect the necessity to replace the inner product in $\mathcal{X}_+$, $\mathcal{X}_0$ or $\mathcal{X}_-$ by an equivalent inner product in order to obtain a quasi Gelfand triple.

Although the previous example suggests to change the inner product in the "almost" pivot space $\mathcal{X}_0$, in our approach it is more convenient to change the inner product (norm) in $\mathcal{X}_-$. In the case of Example A.4 this would mean that we replace $\mathcal{X}_-$ with $\mathrm{L}^2\big((0,1), \frac{1}{wv} \,\mathrm{d}\lambda\big)$, which is the same space with an equivalent inner product (norm).

Lemma A.5. *Let $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ be a triplet of spaces and $D = \operatorname{ran} k_+ \cap \operatorname{ran} k_-$. Then the continuous extension of*

$$\|g\|_{\widehat{\mathcal{X}}_-} = \sup_{f \in k_+^{-1}(D)\setminus\{0\}} \frac{|\langle k_- g, k_+ f\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}}$$

is an equivalent norm on $\mathcal{X}_-$. In particular $\|g\|_{\widehat{\mathcal{X}}_-}$ equals the operator norm of $B(g,\cdot)$ for all $g \in \mathcal{X}_-$.

Proof. By the definition of B and the density of $k_+^{-1}(D)$ in $\mathcal{X}_+$ we have for $g \in k_-^{-1}(D)$

$$\|g\|_{\widehat{\mathcal{X}}_-} = \sup_{f \in k_+^{-1}(D)\setminus\{0\}} \frac{|B(g,f)|}{\|f\|_{\mathcal{X}_+}} = \sup_{f \in \mathcal{X}_+\setminus\{0\}} \frac{|B(g,f)|}{\|f\|_{\mathcal{X}_+}} = \|B(g,\cdot)\|.$$

Consequently, $\mathcal{X}_- \ni g \mapsto \|B(g,\cdot)\|$ is indeed the continuous extension of $k_-^{-1}(D) \ni g \mapsto \|g\|_{\widehat{\mathcal{X}}_-}$. Note that since $B(g,\cdot)$ is a bounded antilinear mapping there exists an $h_g \in \mathcal{X}_+$ such that $B(g,\cdot) = \langle h_g, \cdot\rangle_{\mathcal{X}_+}$. Hence, $\|g\|_{\widehat{\mathcal{X}}_-} = \|h_g\|_{\mathcal{X}_+} \overset{\text{(tos2)}}{\leq} \|g\|_{\mathcal{X}_-}$ and the mapping $g \mapsto h_g$ is bounded. Since by item (tos3) this mapping is also bijective, the open mapping theorem implies that is is boundedly invertible. Therefore, $\|h_g\|_{\mathcal{X}_+} \geq c\|g\|_{\widehat{\mathcal{X}}_-}$ for some $c > 0$, which leads to

$$c\|g\|_{\mathcal{X}_-} \leq \|h_g\|_{\mathcal{X}_+} = \|g\|_{\widehat{\mathcal{X}}_-} = \|h_g\|_{\mathcal{X}_+} \leq \|g\|_{\mathcal{X}_-}.$$ ❑

Lemma A.6. *If $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is triplet of spaces, then $(\mathcal{X}_+, \mathcal{X}_0, \widehat{\mathcal{X}}_-)$ is a quasi Gelfand triple, where $\widehat{\mathcal{X}}_- = \mathcal{X}_-$ equipped with the equivalent norm $\|\cdot\|_{\widehat{\mathcal{X}}_-}$ from Lemma A.5.*

Proof. For $g \in \widehat{\mathcal{X}}_-$ and $f \in \mathcal{X}_+$ we define

$$\langle g, f\rangle_{\widehat{\mathcal{X}}_-,\mathcal{X}_+} := B(g, f).$$

By the definition of the norm of $\widehat{\mathcal{X}}_-$ and (tos3) this is a dual pairing for $(\mathcal{X}_+, \widehat{\mathcal{X}}_-)$ and therefore $(\mathcal{X}_+, \widehat{\mathcal{X}}_-)$ is a dual pair. Note that $\|\cdot\|_{\widehat{\mathcal{X}}_-}$ is equivalent to $\|\cdot\|_{\mathcal{X}_-}$ by Lemma A.5.

We will apply Proposition 3.9 on the embedding $\tilde{\iota}_+ = k_+$ to show that this mapping is closable: Corresponding to $\tilde{\iota}_+$ there is D_- (Definition 3.3), which is a superset of D and therefore dense in $\mathcal{X}_0$ and $\widehat{\mathcal{X}}_-$. Hence, item (iv) of Proposition 3.9 is satisfied and consequently k_+ is closeable. Hence, we define $\iota_+ = \overline{k_+}$ and $\iota_- = (\iota_+^{-1})^*$, where the adjoint is taken w.r.t. the dual pairs $(\mathcal{X}_+, \widehat{\mathcal{X}}_-)$ and $(\mathcal{X}_0, \mathcal{X}_0)$. Note that ι_- is an extension of k_-.[2] For $f \in \operatorname{dom} \iota_+$ and $g \in \operatorname{dom} \iota_-$ we have

$$\langle g, f\rangle_{\widehat{\mathcal{X}}_-,\mathcal{X}_+} = \langle g, \iota_+^{-1}\iota_+ f\rangle_{\widehat{\mathcal{X}}_-,\mathcal{X}_+} = \langle (\iota_+^{-1})^* g, \iota_+ f\rangle_{\mathcal{X}_0} = \langle \iota_- g, \iota_+ f\rangle_{\mathcal{X}_0}.$$

Finally, $\operatorname{dom} \iota_+^* = \operatorname{ran} \iota_-$ holds true by construction of ι_-, which implies that $(\mathcal{X}_+, \mathcal{X}_0, \widehat{\mathcal{X}}_-)$ is a quasi Gelfand triple. ❑

Theorem A.7. *Let $\mathcal{X}_+$, $\mathcal{X}_0$, $\mathcal{X}_-$ be Hilbert spaces. Then $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a triplet of spaces, if and only if $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a quasi Gelfand triple (up to an equivalent norm on $\mathcal{X}_-$)*

Proof. This is the result of Lemmas A.3 and A.6. ❑

Definition A.8 (Triplets of closely embedded Hilbert spaces). Let $\mathcal{X}_+$, $\mathcal{X}_0$, $\mathcal{X}_-$ be Hilbert spaces. Then we say $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ forms a triplet of closely embedded Hilbert spaces, if the following conditions are satisfied.

(th1) There exists a linear operator $j_+ \colon \operatorname{dom} j_+ \subseteq \mathcal{X}_+ \to \mathcal{X}_0$ that is densely defined, injective, closed and $\operatorname{ran} j_+$ is dense in $\mathcal{X}_0$.

(th2) There exists a linear operator $j_- \colon \operatorname{dom} j_- \subseteq \mathcal{X}_0 \to \mathcal{X}_-$ that is densely defined, injective, closed and $\operatorname{ran} j_-$ is dense in $\mathcal{X}_-$.[3]

(th3) $\operatorname{dom} j_+^* \subseteq \operatorname{dom} j_-$ and for every $h \in \operatorname{dom} j_- \subseteq \mathcal{X}_0$ we have

$$\|j_- h\|_{\mathcal{X}_-} = \sup\left\{ \frac{|\langle j_+ f, h\rangle_{\mathcal{X}_0}|}{\|f\|_{\mathcal{X}_+}} \;\middle|\; f \in \operatorname{dom} j_+, f \neq 0 \right\}.$$

Proposition A.9. *Let $\mathcal{X}_+$, $\mathcal{X}_0$, $\mathcal{X}_-$ be Hilbert spaces. Then $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a triplet of closely embedded Hilbert spaces, if and only if $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a quasi Gelfand triple.*

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[2] We did not just regard $\overline{k_-}$, because we do not know whether Conjecture 6.8 holds.

[3] Note that here j_- maps in the reversed direction compared to ι_- in the definition of quasi Gelfand triples.

Proof. Let $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ be a triplet of closely embedded Hilbert spaces. Then we define $\iota_+ := j_+$ and $\iota_- := j_-^{-1}$. By item (th3) we have

$$|\langle \iota_- g, \iota_+ f\rangle_{\mathcal{X}_0}| \leq \|g\|_{\mathcal{X}_-} \|f\|_{\mathcal{X}_+} \quad \text{for} \quad f \in \operatorname{dom} \iota_+, g \in \operatorname{dom} \iota_-.$$

Hence, we can extend the sequilinear form $\langle g, f\rangle_{\mathcal{X}_-,\mathcal{X}_+} := \langle \iota_- g, \iota_+ f\rangle_{\mathcal{X}_0}$ by continuity to $\mathcal{X}_- \times \mathcal{X}_+$. This sequilinear form is a dual pairing of $\mathcal{X}_+$ and $\mathcal{X}_-$ which leads to $(\mathcal{X}_+, \mathcal{X}_-)$ is a dual pair and $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a quasi Gelfand triple.

Let $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ be a quasi Gelfand triple. Then we define $j_+ := \iota_+$ and $j_- := \iota_-^{-1}$ and we immediately obtain that $(\mathcal{X}_+, \mathcal{X}_0, \mathcal{X}_-)$ is a triplet of closely embedded Hilbert spaces. ❑

Appendix B. Auxiliary results

The next lemma is also true for general linear relations. However, since densely defined linear operators are enough for our purpose we restrict ourselves to these operators, also to use commonly known techniques.

Lemma B.1. *Let (X_1, Y_1), (X_1, Z_1), (X_2, Y_2) and (X_2, Z_2) be dual pairs and $\Psi_1 \colon Y_1 \to Z_1$ and $\Psi_2 \colon Y_2 \to Z_2$ be the isomorphisms between Y_1 and Z_1, and Y_2 and Z_2, respectively. Then for a densely defined linear operator A from X_1 to X_2 we have (as illustrated in Figure 6)*

$$A^{*_{Z_2 \times Z_1}} = \Psi_1 A^{*_{Y_2 \times Y_1}} \Psi_2^{-1}.$$

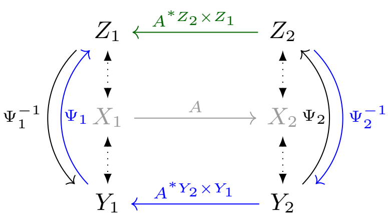

Figure 6. $A^{*_{Z_2 \times Z_1}} = \Psi_1 A^{*_{Y_2 \times Y_1}} \Psi_2^{-1}$

Proof. Let $z_2 \in Z_2$ be such that $\Psi_2^{-1} z \in \operatorname{dom} A^{*_{Y_2 \times Y_1}}$. Then

$$\begin{aligned}\langle Ax_1, z_2\rangle_{X_2,Z_2} &= \langle Ax_1, \Psi_2^{-1} z_2\rangle_{X_2,Y_2} \\ &= \langle x_1, A^{*_{Y_2 \times Y_1}} \Psi_2^{-1} z_2\rangle_{X_1,Y_1} \\ &= \langle x_1, \Psi_1 A^{*_{Y_2 \times Y_1}} \Psi_2^{-1} z_2\rangle_{X_1,Z_1}.\end{aligned}$$

This implies $\Psi_1 A^{*_{Y_2 \times Y_1}} \Psi_2^{-1} \subseteq A^{*_{Z_2 \times Z_1}}$. The same steps with Z_2 and Z_1 replaced with Y_2 and Y_1 yield the reversed inclusion. ❑

The next result can be found in [11, Prop. 3.18] for operators or in [1, Lem. 1.5.8] for linear relations.

Theorem B.2 (J. von Neumann). *Let T be a closed linear operator from the Hilbert space X to the Hilbert space Y. Then T^*T and TT^* are self-adjoint, and $(\mathrm{I}_X + T^*T)$ and $(\mathrm{I}_Y + TT^*)$ are boundedly invertible.*

Note that here the adjoint T^* is calculated with respect to the "natural" dual pairs (X, X) and (Y, Y), i.e., $T^* = T^{*_{Y\times X}}$.

Lemma B.3. *Let T be the operator from the previous theorem. Then* $\operatorname{dom} T^*T$ *is a core of T.*

A proof can be found in [11, Prop. 3.18].

In the next proposition we will look at the situation where we deal with Hilbert spaces, but work with another dual pair. We will denote the adjoint with respect to the canonical Hilbert space dual pair by $*_\mathrm{h}$ and the adjoint with respect to the other dual pair by $*_\mathrm{d}$.

Proposition B.4. *Let X, H be Hilbert spaces, (X, Y) be a complete dual pair and $T\colon \operatorname{dom} T \subseteq X \to H$ be a densely defined and closed linear operator. Then $T^{*_\mathrm{d}}T\colon \operatorname{dom} T^{*_\mathrm{d}}T \subseteq X \to Y$ is self-adjoint, i.e., $(T^{*_\mathrm{d}}T)^{*_\mathrm{d}} = T^{*_\mathrm{d}}T$. Moreover,* $\operatorname{dom} T^{*_\mathrm{d}}T$ *is a core of T.*

Proof. For $x, y \in \operatorname{dom} T^{*_\mathrm{d}}T$ we have

$$\langle T^{*_\mathrm{d}}Tx, y\rangle_{Y,X} = \langle Tx, Ty\rangle_H = \langle x, T^{*_\mathrm{d}}Ty\rangle_{X,Y},$$

which leads to $T^{*_\mathrm{d}}T \subseteq (T^{*_\mathrm{d}}T)^{*_\mathrm{d}}$.

By Theorem B.2 we already know that $T^{*_\mathrm{h}}T$ is self-adjoint. Let $\Psi\colon X \to Y$ the duality mapping, i.e., $\langle \Psi x, y\rangle_{Y,X} = \langle x, y\rangle_X$ for $x, y \in X$. Then $T^{*_\mathrm{d}} = \Psi T^{*_\mathrm{h}}$ (by Lemma B.1) and therefore $T^{*_\mathrm{d}}T = \Psi T^{*_\mathrm{h}}T$. Now for $x \in \operatorname{dom}(T^{*_\mathrm{d}}T)^{*_\mathrm{d}}$ and $y \in \operatorname{dom} T^{*_\mathrm{d}}T = \operatorname{dom} T^{*_\mathrm{h}}T$ we have

$$\begin{aligned}\langle \Psi^{-1}(T^{*_\mathrm{d}}T)^{*_\mathrm{d}}x, y\rangle_X &= \langle (T^{*_\mathrm{d}}T)^{*_\mathrm{d}}x, y\rangle_{Y,X} = \langle x, T^{*_\mathrm{d}}Ty\rangle_{X,Y} \\ &= \langle x, \underbrace{\Psi^{-1}T^{*_\mathrm{d}}}_{=T^{*_\mathrm{h}}}Ty\rangle_X = \langle x, T^{*_\mathrm{h}}Ty\rangle_X.\end{aligned}$$

This implies $\Psi^{-1}(T^{*_\mathrm{d}}T)^{*_\mathrm{d}} \subseteq (T^{*_\mathrm{h}}T)^{*_\mathrm{h}} = T^{*_\mathrm{h}}T$ and applying Ψ on both sides gives $(T^{*_\mathrm{d}}T)^{*_\mathrm{d}} \subseteq \Psi T^{*_\mathrm{h}}T = T^{*_\mathrm{d}}T$.

The last assertion follows from $\operatorname{dom} T^{*_\mathrm{d}}T = \operatorname{dom} T^{*_\mathrm{h}}T$ and $\operatorname{dom} T^{*_\mathrm{h}}T$ is a core of T, ❑

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