

WEAK EQUALS STRONG L^2 REGULARITY FOR PARTIAL TANGENTIAL TRACES ON LIPSCHITZ DOMAINS

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ABSTRACT. We investigate the boundary trace operators that naturally correspond to $H(\text{curl}, \Omega)$, namely the tangential and twisted tangential trace, where $\Omega \subseteq \mathbb{R}^3$. In particular we regard partial tangential traces, i.e., we look only on a subset Γ of the boundary $\partial\Omega$. We assume both Ω and Γ to be strongly Lipschitz (possibly unbounded). We define the space of all $H(\text{curl}, \Omega)$ fields that possess a L^2 tangential trace in a weak sense and show that the set of all smooth fields is dense in that space, which is a generalization of [BBBCD97]. This is especially important for Maxwell's equation with mixed boundary condition as we answer the open problem by Weiss and Staffans in [WS13, Sec. 5] for strongly Lipschitz pairs.

1. INTRODUCTION

We will regard a strongly Lipschitz domain $\Omega \subseteq \mathbb{R}^3$ and the Sobolev space that corresponds to the curl operator

$$H(\text{curl}, \Omega) = \{f \in L^2(\Omega) \mid \text{curl } f \in L^2(\Omega)\}$$

and the “natural” boundary traces that are associated with the curl operator

$$\pi_\tau f := (\nu \times f|_{\partial\Omega}) \times \nu \quad \text{and} \quad \gamma_\tau f := \nu \times f|_{\partial\Omega} \quad \text{for } f \in \mathring{C}^\infty(\mathbb{R}^3),$$

where ν denotes the outer normal vector on the boundary of Ω . These boundary traces are called *tangential trace* and *twisted tangential trace*, respectively. They are motivated by the integration by parts formula

$$\langle \text{curl } f, g \rangle_{L^2(\Omega)} - \langle f, \text{curl } g \rangle_{L^2(\Omega)} = \langle \gamma_\tau f, \pi_\tau g \rangle_{L^2(\partial\Omega)}.$$

We could even extend these boundary operators to $H(\text{curl}, \Omega)$ by introducing suitable boundary spaces, see e.g., [BCS02] for full boundary traces or [Skr21] for partial boundary traces. However, in this article we focus on those $f \in H(\text{curl}, \Omega)$ that have a meaningful $L^2(\partial\Omega)$ (twisted) tangential trace. Hence, for $\Gamma \subseteq \partial\Omega$ we are interested in the following spaces

$$\begin{aligned} \mathring{H}_\Gamma(\text{curl}, \Omega) &= \{f \in H(\text{curl}, \Omega) \mid \pi_\tau f = 0 \text{ on } \Gamma\}, \\ \hat{H}_\Gamma(\text{curl}, \Omega) &= \{f \in H(\text{curl}, \Omega) \mid \pi_\tau f \text{ is in } L^2(\Gamma)\}. \end{aligned}$$

where we will later state precisely what we mean by $\pi_\tau f = 0$ on Γ and $\pi_\tau f \in L^2(\Gamma)$. In particular we are interested in $\hat{H}_\Gamma(\text{curl}, \Omega)$. Similar to Sobolev spaces there are two approaches to $\pi_\tau f \in L^2(\Gamma)$: A weak approach by representation in an inner product and a strong approach by limits of regular functions. We use the weak approach as definition, see Definition 4.1. The question that immediately arises is: “Do both approaches lead to the same space?”

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In [WS13, eq. (5.20)] the authors observed this problem and concluded that it can cause ambiguity for boundary conditions, if the approaches don't coincide. In fact they stated this issue at the end of section 5 in [WS13] as an open problem. This problem can actually be viewed as a more general question that arises for quasi Gelfand triples, see [Skr23b, Conjecture 6.7].

We will not explicitly define the strong approach, but show that the most regular functions (C^∞ functions) are already dense in the weakly defined space, which immediately implies that any strong approach with less regular functions (e.g., H^1) will lead to the same space. This is exactly what was done in [BBBCD97] for bounded Ω with $\Gamma = \partial\Omega$. Hence, we present a generalization of [BBBCD97] for partial L^2 tangential traces. In particular, we aim to prove the following two main theorems.

Theorem 1.1. *Let Ω be a (possibly unbounded) strongly Lipschitz domain and $\Gamma_1 \subseteq \partial\Omega$ such that (Ω, Γ_1) is a strongly Lipschitz pair, then $\dot{C}^\infty(\mathbb{R}^3)$ is dense in $\dot{H}_{\Gamma_1}(\text{curl}, \Omega)$ with respect to $\|\cdot\|_{\dot{H}_{\Gamma_1}(\text{curl}, \Omega)}$.*

Theorem 1.2. *Let Ω be a (possibly unbounded) strongly Lipschitz domain and $\Gamma_0 \subseteq \partial\Omega$ such that (Ω, Γ_0) is a strongly Lipschitz pair, then $\dot{C}_{\Gamma_0}^\infty(\mathbb{R}^3)$ is dense in $\dot{H}_{\partial\Omega}(\text{curl}, \Omega) \cap \dot{H}_{\Gamma_0}(\text{curl}, \Omega)$ with respect to $\|\cdot\|_{\dot{H}_{\partial\Omega}(\text{curl}, \Omega)}$.*

However, it turned out that it is best to prove them in reversed order.

The importance of our density results lies in the context of Maxwell's equations with boundary conditions that involve a mixture of π_τ and γ_τ in the sense of linear combination, e.g., this simplified instance of Maxwell's equations

$$\begin{aligned} \partial_t E(t, \zeta) &= \text{curl } H(t, \zeta), & t \geq 0, \zeta \in \Omega, \\ \partial_t H(t, \zeta) &= -\text{curl } E(t, \zeta), & t \geq 0, \zeta \in \Omega, \\ \pi_\tau E(t, \xi) + \gamma_\tau H(t, \xi) &= 0, & t \geq 0, \xi \in \Gamma_1, \\ \pi_\tau E(t, \xi) &= 0, & t \geq 0, \xi \in \Gamma_0. \end{aligned}$$

In order to properly formulate the boundary conditions we need to know what functions E, H have tangential traces that allow such a linear combination. Especially when it comes to well-posedness our density results are needed to avoid the ambiguity that was observed in [WS13].

As suspected by Weiss and Staffans in [WS13] the regularity of the interface of $\Gamma_0 \subseteq \partial\Omega$ and $\Gamma_1 := \partial\Omega \setminus \overline{\Gamma_0}$ seems to play a role. At least for our answer we need that the boundary of Γ_0 is also strongly Lipschitz.

In particular our strategy is based on the following decomposition from [PS22a, Thm. 5.2] for bounded Ω

$$\dot{H}_{\Gamma_0}(\text{curl}, \Omega) = \dot{H}_{\Gamma_0}^1(\Omega) + \nabla \dot{H}_{\Gamma_0}^1(\Omega), \quad (1)$$

which requires (Ω, Γ_0) to be a strongly Lipschitz pair. Every element of $\dot{H}_{\Gamma_0}^1(\Omega)$ can be approximated by a sequence in $\dot{C}_{\Gamma_0}^\infty(\mathbb{R}^3)$ w.r.t. $\|\cdot\|_{H^1(\Omega)}$ (see [BPS16, Lem. 3.1]), which is a stronger norm than the “natural” norm of $\dot{H}_{\partial\Omega}(\text{curl}, \Omega)$. Hence, the challenging part will be finding an approximation by $\dot{C}_{\Gamma_0}^\infty(\mathbb{R}^3)$ elements for all elements in

$$\dot{H}_{\partial\Omega}(\text{curl}, \Omega) \cap \nabla \dot{H}_{\Gamma_0}^1(\Omega).$$

It even turned out that, if we can prove the decomposition (1) also for less regular Γ_0 , then our main theorems would automatically generalize for those less regular partitions of $\partial\Omega$, since this is the only occasion where the regularity of Γ_0 is used.

Finally, we will conclude the results for unbounded Ω by localization.

2. PRELIMINARY

For $\Omega \subseteq \mathbb{R}^d$ open and $\Gamma \subseteq \partial\Omega$ open we use the following notation (as in [BPS16])

$$\begin{aligned}\mathring{C}^\infty(\Omega) &:= \{f \in C^\infty(\Omega) \mid \text{supp } f \text{ is compact in } \Omega\}, \\ \mathring{C}_\Gamma^\infty(\Omega) &:= \left\{f|_\Omega \mid f \in \mathring{C}^\infty(\mathbb{R}^d), \text{dist}(\Gamma, \text{supp } f) > 0\right\},\end{aligned}$$

and $H^1(\Omega)$ denotes the usual Sobolev space and $\mathring{H}_\Gamma^1(\Omega)$ is the subspace of $H^1(\Omega)$ with homogeneous boundary data on Γ , i.e., $\mathring{H}_\Gamma^1(\Omega) = \overline{\mathring{C}_\Gamma^\infty(\Omega)}^{H^1(\Omega)}$.

Note that the trace operators π_τ and γ_τ are called tangential traces, because $\nu \cdot \pi_\tau f = 0$ and $\nu \cdot \gamma_\tau f = 0$. Hence, it is natural to introduce the *tangential* L^2 space on $\Gamma \subseteq \partial\Omega$ by

$$L_\tau^2(\Gamma) = \{f \in L^2(\Gamma) \mid \nu \cdot f = 0\}.$$

This space is again a Hilbert space with the $L^2(\Gamma)$ inner product. Moreover, both $\pi_\tau \mathring{C}_{\partial\Omega \setminus \Gamma}^\infty(\mathbb{R}^3)$ and $\gamma_\tau \mathring{C}_{\partial\Omega \setminus \Gamma}^\infty(\mathbb{R}^3)$ are dense in that space.

Next we recall the definition of a strongly Lipschitz domain, see e.g., [Gri85]. Moreover, we need H^1 spaces on strongly Lipschitz boundaries, see e.g., [Skr23a] for a careful treatment.

Definition 2.1. Let Ω be an open subset of \mathbb{R}^d . We say Ω is a *strongly Lipschitz domain*, if for every $p \in \partial\Omega$ there exist $\epsilon, h > 0$, a hyperplane $W = \text{span}\{w_1, \dots, w_{d-1}\}$, where $\{w_1, \dots, w_{d-1}\}$ is an orthonormal basis of W , and a Lipschitz continuous function $a: (p + W) \cap B_\epsilon(p) \rightarrow (-\frac{h}{2}, \frac{h}{2})$ such that

$$\begin{aligned}\partial\Omega \cap C_{\epsilon,h}(p) &= \{x + a(x)v \mid x \in (p + W) \cap B_\epsilon(p)\}, \\ \Omega \cap C_{\epsilon,h}(p) &= \{x + sv \mid x \in (p + W) \cap B_\epsilon(p), -h < s < a(x)\},\end{aligned}$$

where v is the normal vector of W and $C_{\epsilon,h}(p)$ is the cylinder $\{x + \delta v \mid x \in (p + W) \cap B_\epsilon(p), \delta \in (-h, h)\}$.

The boundary $\partial\Omega$ is then called *strongly Lipschitz boundary*.

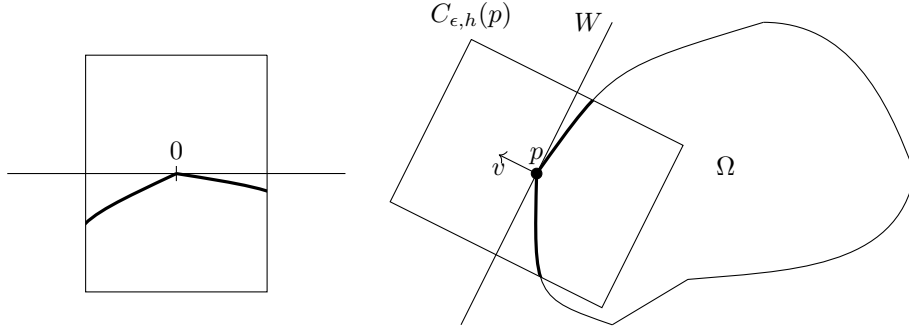


FIGURE 1. Lipschitz boundary

Corresponding to a strongly Lipschitz domain we define the following bi-Lipschitz continuous mapping

$$k: \begin{cases} \partial\Omega \cap C_{\epsilon,h}(p) & \rightarrow B_\epsilon(0) \subseteq \mathbb{R}^{d-1}, \\ \zeta & \mapsto W^T(\zeta - p), \end{cases}$$

where we used W as the matrix $[w_1 \dots w_{d-1}]$. We call this mapping a *strongly Lipschitz chart* of $\partial\Omega$ and we call its domain the *chart domain*. Its inverse is given

by

$$k^{-1}: \begin{cases} B_\epsilon(0) \subseteq \mathbb{R}^{d-1} & \rightarrow \partial\Omega \cap C_{\epsilon,h}(p), \\ x & \mapsto p + Wx + a(x)v, \end{cases}$$

where we will use $a(x)$ also as shortcut for $a(p + Wx)$, which is then a Lipschitz continuous function from $B_\epsilon(0) \subseteq \mathbb{R}^{d-1}$ to \mathbb{R} . Charts are used to regard the surface of Ω locally as a flat subset of \mathbb{R}^{d-1} . Every restriction of a chart k to an open $\Gamma \subseteq \partial\Omega$ is again a chart.

Definition 2.2. Let Ω be a strongly Lipschitz domain in \mathbb{R}^d . Then we say that an open $\Gamma_0 \subseteq \partial\Omega$ is strongly Lipschitz, if for every $p \in \overline{\Gamma_0}$ there exists a chart $k: \partial\Omega \cap C_{\epsilon,h}(p) \rightarrow B_\epsilon(0) \subseteq \mathbb{R}^{d-1}$ such that $k(\Gamma_0)$ is a strongly Lipschitz domain in \mathbb{R}^{d-1} .

The boundary $\partial\Gamma_0$ is then called *strongly Lipschitz boundary*.

Note that it is sufficient to reduce the condition in the previous definition to $p \in \partial\Gamma_0$ instead of $p \in \overline{\Gamma_0}$.

Definition 2.3. We call (Ω, Γ_0) a *strongly Lipschitz pair*, if Ω is a strongly Lipschitz domain and $\Gamma_0 \subseteq \partial\Omega$ is strongly Lipschitz.

Note that if $\Gamma_0 \subseteq \partial\Omega$ is strongly Lipschitz, then also $\Gamma_1 := \partial\Omega \setminus \overline{\Gamma_0}$ is strongly Lipschitz. Hence, if (Ω, Γ_0) is a strongly Lipschitz pair, then also (Ω, Γ_1) is.

Since we only deal with strongly Lipschitz domains and boundaries, we will omit the term “strongly” and just say *Lipschitz domain*, *Lipschitz boundary* and *Lipschitz chart*.

Recall the definition of a H^1 function on the boundary of a Lipschitz domain, see e.g., [Skr23a]. Since we only need the following for bounded domains we will state the following definition only for bounded domains.

Definition 2.4. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. We say $f \in L^2(\partial\Omega)$ is in $H^1(\partial\Omega)$, if for every Lipschitz chart $k: \Gamma \rightarrow U$ the mapping

$$f \circ k^{-1} \text{ is in } H^1(U).$$

3. DENSITY RESULTS FOR $W(\Omega)$

In this section we restrict ourselves to bounded Lipschitz domains Ω . However, this is mostly for convenience.

Definition 3.1. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. Then we define

$$W(\Omega) := \{f \in H^1(\Omega) \mid f|_{\partial\Omega} \in H^1(\partial\Omega)\},$$

$$\|f\|_{W(\Omega)} := \left(\|f\|_{H^1(\Omega)}^2 + \|f|_{\partial\Omega}\|_{H^1(\partial\Omega)}^2 \right)^{1/2}.$$

The next lemma is a crucial tool in our construction. The basic idea is: Take a smooth function with compact support on a flat domain ($U \subseteq \mathbb{R}^{d-1}$) extend it on the entire hyperplane \mathbb{R}^{d-1} by 0, and then extend it constantly in the orthogonal direction, i.e., $f(\zeta + \lambda e_d) = f(\zeta)$, where $\lambda \in \mathbb{R}$ and e_d is the d -th unit vector. A multiplication with a cutoff function makes sure that this extension has compact support. By rotation and translation this can be done for arbitrary hyperplanes. Figure 2 illustrates the construction.

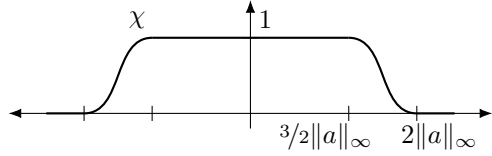
Lemma 3.2. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain, $\Gamma \subseteq \partial\Omega$, $k: \Gamma \rightarrow U$ be a Lipschitz chart and $f \in H^1(\partial\Omega)$ with compact support in $\Gamma' \subseteq \Gamma$. Then there exists an $F \in H^1(\mathbb{R}^d) \cap W(\Omega) \cap \dot{H}_{\partial\Omega \setminus \Gamma'}^1(\Omega)$ such that $F|_{\partial\Omega} = f$. Moreover, there exists

a sequence $(F_n)_{n \in \mathbb{N}}$ in $\dot{C}_{\partial\Omega \setminus \Gamma'}^\infty(\mathbb{R}^d)$ that converges to F w.r.t. $\|\cdot\|_{H^1(\mathbb{R}^d)} + \|\cdot\|_{W(\Omega)}$, i.e., F_n converges to F in $H^1(\mathbb{R}^d)$ and $F_n|_{\partial\Omega}$ converges to $F|_{\partial\Omega}$ in $H^1(\partial\Omega)$.

Proof. Let p , W and v be the point, hyperplane and normal vector, respectively, to the chart k . In particular k^{-1} is given by

$$k^{-1}: \begin{cases} U \subseteq \mathbb{R}^{d-1} & \rightarrow \Gamma, \\ x & \mapsto p + Wx + a(x)v, \end{cases}$$

where U is open and a is the Lipschitz function. Let $\chi \in \dot{C}^\infty(\mathbb{R})$ be a cut-off function such that

$$\chi(\lambda) \in \begin{cases} \{1\}, & |\lambda| < 3/2\|a\|_\infty, \\ [0, 1], & |\lambda| \in (3/2, 2)\|a\|_\infty, \\ \{0\}, & |\lambda| > 2\|a\|_\infty. \end{cases}$$


By definition $\hat{f} = f \circ k^{-1}$ is in $H^1(U)$ and since f has compact support in Γ' we conclude $\hat{f} \in \dot{H}^1(U)$ with support in $U' := k(\Gamma')$. Note that we can extend $\hat{f} \in \dot{H}^1(U)$ on \mathbb{R}^d by 0. We define

$$F(\zeta) = \chi(v \cdot (\zeta - p)) \hat{f}(W^T(\zeta - p)) \quad \text{for } \zeta \in \mathbb{R}^d$$

The support of F is inside of a rotated and translated version of $U' \times \text{supp } \chi$, in particular

$$\text{supp } F \subseteq p + [W \quad v] (U' \times \text{supp } \chi) =: \Xi.$$

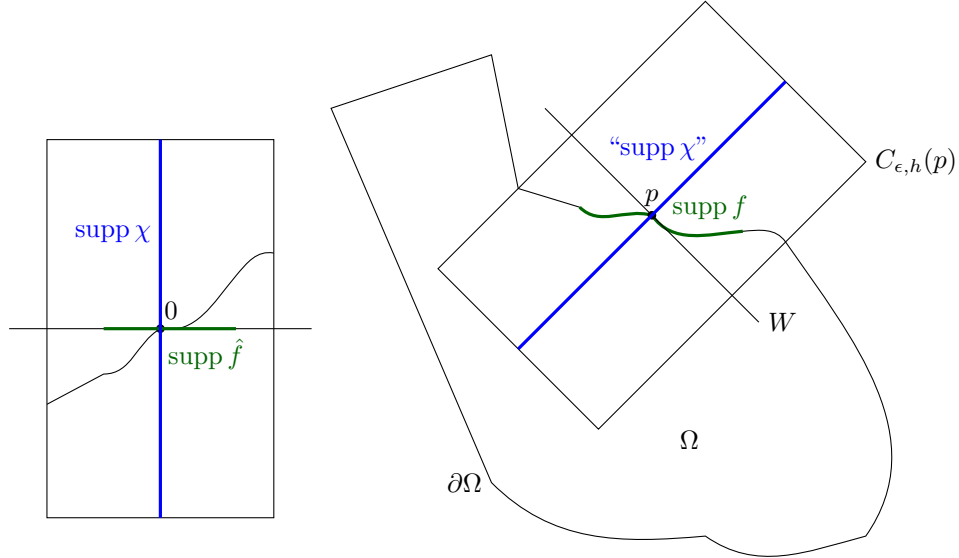


FIGURE 2. Illustration of the construction of Lemma 3.2

Note that by construction of χ we have $\text{supp } F \cap \partial\Omega \subseteq \Gamma'$, therefore $F|_{\partial\Omega \setminus \Gamma'} = 0$. Since $\hat{f} \in H^1(\mathbb{R}^{d-1})$ it is straight forward that F possess $L^2(\mathbb{R}^d)$ directional derivatives in W directions. Moreover, by construction (and the Leibniz product rule) $\frac{\partial}{\partial v} F = \chi' \hat{f}(W^T(\cdot - p))$, which implies $F \in H^1(\mathbb{R}^d)$. By definition of a Lipschitz chart we have $|v \cdot (\zeta - p)| \leq \|a\|_\infty$ for $\zeta \in \Gamma$ and hence

$$F(\zeta) = \underbrace{\chi(v \cdot (\zeta - p))}_{=1} \hat{f}(W^T(\zeta - p)) = \hat{f} \circ k(\zeta) = f(\zeta) \quad \text{for } \zeta \in \Gamma$$

(a.e. w.r.t. the surface measure).

By assumption on \hat{f} there exists a sequence $(\hat{f}_n)_{n \in \mathbb{N}}$ in $\dot{C}^\infty(U)$ that converges to \hat{f} w.r.t. $\|\cdot\|_{H^1(U)}$. Note that \hat{f}_n is also in $\dot{C}^\infty(\mathbb{R}^{d-1})$. We define

$$F_n(\zeta) = \chi(v \cdot (\zeta - p)) \hat{f}_n(W^\top(\zeta - p)) \quad \text{for } \zeta \in \mathbb{R}^d$$

Note that F_n is the composition of C^∞ mappings and therefore in $C^\infty(\mathbb{R}^d)$. Again, the support of F_n is contained in the bounded set Ξ and therefore compact, which implies $F_n \in \dot{C}^\infty(\mathbb{R}^d)$. Note that $F_n \circ k^{-1} = \hat{f}_n$, which implies $(F_n \circ k^{-1})_{n \in \mathbb{N}}$ converges to \hat{f} w.r.t. $\|\cdot\|_{H^1(U)}$. Since $F_n|_{\partial\Omega \setminus \Gamma} = 0 = F|_{\partial\Omega \setminus \Gamma}$ we conclude $F_n|_{\partial\Omega} \rightarrow F|_{\partial\Omega}$ in $H^1(\partial\Omega)$. Finally,

$$\begin{aligned} \|F_n - F\|_{H^1(\mathbb{R}^3)} &\leq \|\chi'\|_\infty \|(\hat{f}_n - \hat{f})(W^\top(\cdot - p))\|_{H^1(\Xi)} \\ &\leq 2\|a\|_\infty \|\chi'\|_\infty \|\hat{f}_n - \hat{f}\|_{H^1(U)} \rightarrow 0. \quad \square \end{aligned}$$

We will formulate a generalization of [BBBCD97, 2. Preliminaries].

Theorem 3.3. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain and $\Gamma \subseteq \partial\Omega$. Then $\dot{C}_\Gamma^\infty(\mathbb{R}^d)$ is dense in $W(\Omega) \cap \dot{H}_\Gamma^1(\Omega)$ w.r.t. $\|\cdot\|_{W(\Omega)}$.*

Proof. Since Ω is a Lipschitz domain we find for every $p \in \partial\Omega$ a cylinder $C_{\epsilon,h}(p)$ (ϵ and h depend on p) and a Lipschitz chart $k: \partial\Omega \cap C_{\epsilon,h}(p) \rightarrow B_\epsilon(0) \subseteq \mathbb{R}^{d-1}$.

Hence we can cover $\partial\Omega$ by $\bigcup_{p \in \partial\Omega} C_{\epsilon,h}(p)$ and by compactness of $\partial\Omega$ there are already finitely many p_i , $i \in \{1, \dots, m\}$ such that

$$\partial\Omega \subseteq \bigcup_{i=1}^m \underbrace{C_{\epsilon_i, h_i}(p_i)}_{=: \Omega_i}$$

We employ a partition of unity and obtain $(\alpha_i)_{i=1}^m$, subordinate to this cover, i.e.,

$$\alpha_i \in \dot{C}^\infty(\Omega_i), \quad \alpha_i(\zeta) \in [0, 1], \quad \text{and} \quad \sum_{i=1}^m \alpha_i(\zeta) = 1 \quad \text{for all } \zeta \in \partial\Omega.$$

For $f \in W(\Omega) \cap \dot{H}_\Gamma^1(\Omega)$ we define $f_i := \alpha_i f$. It is straightforward to show $f_i \in W(\Omega) \cap \dot{H}_\Gamma^1(\Omega)$. For every Ω_i there is a Lipschitz chart $k_i: \Gamma_i \rightarrow U_i \subseteq \mathbb{R}^{d-1}$, where $\Gamma_i = \partial\Omega \cap \Omega_i$. Moreover, $f_i|_{\partial\Omega}$ has support in $\Gamma_i \cap \Gamma^c$, where $\Gamma^c = (\partial\Omega \setminus \Gamma)$.

By Lemma 3.2 there is an $F_i \in H^1(\mathbb{R}^d) \cap W(\Omega) \cap \dot{H}_{\partial\Omega \setminus (\Gamma_i \cap \Gamma^c)}^1(\mathbb{R}^d)$ such that $F_i|_{\partial\Omega} = f_i|_{\partial\Omega}$ and a sequence $(F_{i,n})_{n \in \mathbb{N}}$ in $\dot{C}_{\partial\Omega \setminus (\Gamma_i \cap \Gamma^c)}^\infty(\mathbb{R}^d) \subseteq \dot{C}_\Gamma^\infty(\mathbb{R}^d)$ that converges to F_i in $H(\mathbb{R}^d)$ and in $W(\Omega)$. Hence, we have

$$f - \sum_{i=1}^m F_i \in \dot{H}^1(\Omega),$$

which can be approximated by $(F_{0,n})_{n \in \mathbb{N}}$ in $\dot{C}^\infty(\Omega)$. Hence, $(\sum_{i=0}^m F_{i,n})_{n \in \mathbb{N}}$ is a sequence in $\dot{C}_\Gamma^\infty(\mathbb{R}^d)$ and converges to f in $W(\Omega)$. \square

4. THE BOUNDED CASE

4.1. Density result with homogeneous part. In this section we will finally define the Sobolev spaces with homogeneous and L^2 partial tangential traces, respectively. Then we will prove one of the main results for bounded domains.

We will use a weak definition for the tangential trace and twisted tangential trace as, e.g., in [PS22b].

Definition 4.1. Let Ω be a (possibly unbounded) Lipschitz domain and $\Gamma \subseteq \partial\Omega$ open (in $\partial\Omega$).

- We say $f \in H(\text{curl}, \Omega)$ has a $L^2_\tau(\Gamma)$ tangential trace, if there exists a $q \in L^2_\tau(\Gamma)$ such that

$$\langle f, \text{curl} \phi \rangle_{L^2(\Omega)} - \langle \text{curl} f, \phi \rangle_{L^2(\Omega)} = \langle q, \gamma_\tau \phi \rangle_{L^2_\tau(\Gamma)} \quad \forall \phi \in \mathring{C}^\infty_{\partial\Omega \setminus \Gamma}(\mathbb{R}^3).$$

In this case we say $\pi_\tau f \in L^2_\tau(\Gamma)$ and $\pi_\tau f = q$ on Γ or more precisely $\pi_\tau|_\Gamma f = q$.

- We say $f \in H(\text{curl}, \Omega)$ has a $L^2_\tau(\Gamma)$ twisted tangential trace, if there exists a $q \in L^2_\tau(\Gamma)$ such that

$$\langle \text{curl} f, \phi \rangle_{L^2(\Omega)} - \langle f, \text{curl} \phi \rangle_{L^2(\Omega)} = \langle q, \pi_\tau \phi \rangle_{L^2_\tau(\Gamma)} \quad \forall \phi \in \mathring{C}^\infty_{\partial\Omega \setminus \Gamma}(\mathbb{R}^3).$$

In this case we say $\gamma_\tau f \in L^2_\tau(\Gamma)$ and $\gamma_\tau f = q$ on Γ or more precisely $\gamma_\tau|_\Gamma f = q$.

Note that the previous definition does not say anything about $\pi_\tau f$ on $\partial\Omega \setminus \Gamma$. Moreover, the notation $\pi_\tau|_\Gamma$ and $\gamma_\tau|_\Gamma$ is an abuse of notation, which should indicate that we only look at the part Γ of the boundary $\partial\Omega$.

Remark 4.2. Note that $\nu \times \gamma_\tau \phi = -\pi_\tau \phi$ and $\langle q, \gamma_\tau \phi \rangle_{L^2_\tau(\Gamma)} = \langle \nu \times q, \nu \times \gamma_\tau \phi \rangle_{L^2_\tau(\Gamma)}$. Hence, we can easily see that $\pi_\tau f \in L^2_\tau(\Gamma)$ is equivalent to $\gamma_\tau f \in L^2_\tau(\Gamma)$ and $\gamma_\tau f = \nu \times \pi_\tau f$.

Definition 4.3. Let Ω be a (possibly unbounded) Lipschitz domain and $\Gamma \subseteq \partial\Omega$ open (in $\partial\Omega$). Then we define the space

$$\hat{H}_\Gamma(\text{curl}, \Omega) := \{f \in H(\text{curl}, \Omega) \mid \pi_\tau f \in L^2_\tau(\Gamma)\}$$

with its norm

$$\|f\|_{\hat{H}_\Gamma(\text{curl}, \Omega)} := \left(\|f\|_{L^2(\Omega)}^2 + \|\text{curl} f\|_{L^2(\Omega)}^2 + \|\pi_\tau f\|_{L^2_\tau(\Gamma)}^2 \right)^{1/2}.$$

For $\Gamma = \partial\Omega$ we will just write $\hat{H}(\text{curl}, \Omega)$ instead of $\hat{H}_{\partial\Omega}(\text{curl}, \Omega)$.

Definition 4.4. Let Ω be a (possibly unbounded) Lipschitz domain and $\Gamma \subseteq \partial\Omega$ open (in $\partial\Omega$). Then we define the space

$$\mathring{H}_\Gamma(\text{curl}, \Omega) = \{f \in \hat{H}_\Gamma(\text{curl}, \Omega) \mid \pi_\tau|_\Gamma f = 0\}.$$

For $\Gamma = \partial\Omega$ we will just write $\mathring{H}(\text{curl}, \Omega)$ instead of $\mathring{H}_{\partial\Omega}(\text{curl}, \Omega)$.

In [BPS16, Thm. 4.5] it is shown (for bounded Ω) that $\mathring{C}^\infty_\Gamma(\Omega)$ is dense in $\mathring{H}_\Gamma(\text{curl}, \Omega)$ w.r.t. $\|\cdot\|_{H(\text{curl}, \Omega)}$, i.e.,

$$\mathring{H}_\Gamma(\text{curl}, \Omega) = \overline{\mathring{C}^\infty_\Gamma(\Omega)}^{H(\text{curl}, \Omega)}.$$

Hence, for homogeneous tangential traces there is already a version of the desired density result.

Note that the hat on top of the H indicates partial L^2 boundary conditions and the circle on top indicates partial homogeneous boundary conditions.

Remark 4.5. We can immediately see

$$\mathring{H}_\Gamma(\text{curl}, \Omega) \subseteq \hat{H}_\Gamma(\text{curl}, \Omega).$$

Since $\pi_\tau f \in L^2_\tau(\Gamma)$ is equivalent to $\gamma_\tau f \in L^2_\tau(\Gamma)$ we have

$$\hat{H}_\Gamma(\text{curl}, \Omega) = \{f \in H(\text{curl}, \Omega) \mid \gamma_\tau f \in L^2_\tau(\Gamma)\},$$

Since $\pi_\tau f = \gamma_\tau f \times \nu$, we have $\|\pi_\tau f\|_{L^2_\tau(\Gamma)} = \|\gamma_\tau f\|_{L^2_\tau(\Gamma)}$ and

$$\|f\|_{\hat{H}_\Gamma(\text{curl}, \Omega)} = \left(\|f\|_{L^2(\Omega)}^2 + \|\text{curl} f\|_{L^2(\Omega)}^2 + \|\gamma_\tau f\|_{L^2_\tau(\Gamma)}^2 \right)^{1/2}.$$

Remark 4.6. Since we use representation in an inner product, one can say that we have defined $\hat{H}_\Gamma(\text{curl}, \Omega)$ weakly. Another approach could have been to regard $\overline{\dot{C}^\infty(\mathbb{R}^3)}^{\hat{H}_\Gamma(\text{curl}, \Omega)}$, which could be called a strong approach. From this perspective the result we are going to show basically tells us that the weak and the strong approach to $H(\text{curl}, \Omega)$ fields that possess a $L^2_\tau(\Gamma)$ tangential trace coincide.

Recall the decomposition (1) for a Lipschitz pair (Ω, Γ_0) , where Ω is bounded:

$$\dot{H}_{\Gamma_0}(\text{curl}, \Omega) = \dot{H}_{\Gamma_0}^1(\Omega) + \nabla \dot{H}_{\Gamma_0}^1(\Omega).$$

Note that every element in $\dot{H}_{\Gamma_0}^1(\Omega)$ is already in $\hat{H}(\text{curl}, \Omega) \cap \dot{H}_{\Gamma_0}(\text{curl}, \Omega)$. Moreover, by [BPS16, Lem. 3.1] $\dot{C}_{\Gamma_0}^\infty(\mathbb{R}^3)$ is dense in $\dot{H}_{\Gamma_0}^1(\Omega)$ w.r.t. $\|\cdot\|_{H^1(\Omega)}$ and therefore also w.r.t. $\|\cdot\|_{\hat{H}(\text{curl}, \Omega)}$.

Hence, it is left to show that every

$$f \in \nabla \dot{H}_{\Gamma_0}^1(\Omega) \cap \hat{H}(\text{curl}, \Omega)$$

can be approximated by a $\dot{C}_{\Gamma_0}^\infty(\mathbb{R}^3)$ function (w.r.t. $\|\cdot\|_{\hat{H}(\text{curl}, \Omega)}$).

The following result seems obvious at first glance, but needs a bit of preparation to be shown. One must resist the temptation to show it for smooth function and conclude it by density, since this would lead to a circular argument. Luckily, it is basically [Skr23a, Thm. 4.2] and therefore already shown.

Lemma 4.7. *Let Ω be a bounded Lipschitz domain, $\Gamma_0 \subseteq \partial\Omega$ open and $f \in \dot{H}_{\Gamma_0}^1(\Omega)$ such that $\nabla f \in \hat{H}(\text{curl}, \Omega)$ (in particular $\pi_\tau \nabla f \in L^2_\tau(\partial\Omega)$). Then $\pi_\tau \nabla f = \nabla_\tau f|_{\partial\Omega}$ and $f \in W(\Omega) \cap \dot{H}_{\Gamma_0}^1(\Omega)$.*

Proof. Since $\nabla f \in \hat{H}(\text{curl}, \Omega)$, we know that $\pi_\tau \nabla f \in L^2(\partial\Omega)$ which implies $f|_{\partial\Omega} \in H^1(\partial\Omega)$ and $\nabla_\tau f|_{\partial\Omega} = \pi_\tau \nabla f$, see [Skr23a, Thm. 4.2]. Therefore, we conclude $f \in W(\Omega)$. \square

This brings us to our first main theorem.

Theorem 4.8. *Let Ω be bounded and (Ω, Γ_0) a Lipschitz pair. Then $\dot{C}_{\Gamma_0}^\infty(\mathbb{R}^3)$ is dense in $\hat{H}(\text{curl}, \Omega) \cap \dot{H}_{\Gamma_0}(\text{curl}, \Omega)$ w.r.t. $\|\cdot\|_{\hat{H}(\text{curl}, \Omega)}$.*

Proof. Let $f \in \hat{H}(\text{curl}, \Omega) \cap \dot{H}_{\Gamma_0}(\text{curl}, \Omega)$ be arbitrary. Then we can decompose f into $f = f_1 + \nabla f_2$, where $f_1 \in \dot{H}_{\Gamma_0}^1(\Omega)$ and $f_2 \in \dot{H}_{\Gamma_0}^1(\Omega)$. Note that $f \in \hat{H}(\text{curl}, \Omega)$ and $f_1 \in \hat{H}(\text{curl}, \Omega)$, which implies $\nabla f_2 \in \hat{H}(\text{curl}, \Omega) \cap \nabla \dot{H}_{\Gamma_0}^1(\Omega)$.

By [BPS16, Lem. 3.1] f_1 can be approximated by $\dot{C}_{\Gamma_0}^\infty(\mathbb{R}^3)$ functions w.r.t. $\|\cdot\|_{H^1(\Omega)}$ and therefore also w.r.t. $\|\cdot\|_{\hat{H}(\text{curl}, \Omega)}$.

By Lemma 4.7 we know that $f_2 \in W(\Omega) \cap \dot{H}_{\Gamma_0}^1(\Omega)$. Hence, we can apply Theorem 3.3 and obtain a sequence $(f_{2,n})_{n \in \mathbb{N}}$ that converges to f_2 w.r.t. $\|\cdot\|_{W(\Omega)}$. This gives

$$\begin{aligned} & \|\nabla f_2 - \nabla f_{2,n}\|_{\hat{H}(\text{curl}, \Omega)}^2 \\ &= \|\nabla(f_2 - f_{2,n})\|_{L^2(\Omega)}^2 + \underbrace{\|\text{curl} \nabla(f_2 - f_{2,n})\|_{L^2(\Omega)}^2}_{=0} + \|\pi_\tau \nabla(f_2 - f_{2,n})\|_{L^2(\partial\Omega)}^2 \\ &\leq \|f_2 - f_{2,n}\|_{H^1(\Omega)}^2 + \|f_2|_{\partial\Omega} - f_{2,n}|_{\partial\Omega}\|_{H^1(\partial\Omega)}^2 \\ &= \|f_2 - f_{2,n}\|_{W(\Omega)}^2 \rightarrow 0, \end{aligned}$$

which finishes the proof. \square

4.2. Density result without homogeneous part. Since we already know that $\mathring{C}_{\Gamma_0}^\infty(\mathbb{R}^3)$ is dense in $\hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega)$, we can show the density of $\mathring{C}^\infty(\mathbb{R}^3)$ in $\hat{H}_{\Gamma_1}(\text{curl}, \Omega)$ by a duality argument, which we will present in this section. This argument can be done in just a few lines within the notion of quasi Gelfand triples [Skr23b]. However, in order to stay relatively elementary we extract the essence and build a proof that avoids the introduction of this notion.

Basically we mimic the abstract boundary space for the tangential trace by $\hat{H}(\text{curl}, \Omega)^\perp$, which can also be viewed as the boundary space as it is isometrically isomorphic.

Our standing assumption in this section is that Ω is bounded, (Ω, Γ_0) is a Lipschitz pair and $\Gamma_1 := \partial\Omega \setminus \overline{\Gamma_0}$.

Note that the boundedness of Ω is only necessary to apply Theorem 4.8. However, we will later be able to replace this by Theorem 5.5.

Corollary 4.9. *If $f \in \hat{H}_{\Gamma_1}(\text{curl}, \Omega)$, then*

$$\langle \gamma_\tau f, \pi_\tau g \rangle_{L^2(\Gamma_1)} = \langle \text{curl } f, g \rangle_{L^2(\Omega)} - \langle f, \text{curl } g \rangle_{L^2(\Omega)}$$

for all $g \in \hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega)$.

Proof. For $f \in \hat{H}_{\Gamma_1}(\text{curl}, \Omega)$ we have by definition

$$\langle \gamma_\tau f, \pi_\tau g \rangle_{L^2(\Gamma_1)} = \langle \text{curl } f, g \rangle_{L^2(\Omega)} - \langle f, \text{curl } g \rangle_{L^2(\Omega)}$$

for all $g \in \mathring{C}_{\Gamma_0}^\infty(\mathbb{R}^3)$. Since this equation is continuous in g w.r.t. $\|\cdot\|_{\hat{H}(\text{curl}, \Omega)}$,

we can extend it by continuity to $g \in \overline{\mathring{C}_{\Gamma_0}^\infty(\mathbb{R}^3)}^{\hat{H}(\text{curl}, \Omega)}$ and by Theorem 4.8 to $g \in \hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega)$. \square

Lemma 4.10. *We have the following identity*

$$\hat{H}(\text{curl}, \Omega)^\perp = \{f \in \hat{H}(\text{curl}, \Omega) \mid \text{curl } \text{curl } f = -f\},$$

where the orthogonal complement is taken in $\hat{H}(\text{curl}, \Omega)$, i.e., w.r.t. $\langle \cdot, \cdot \rangle_{\hat{H}(\text{curl}, \Omega)}$. Moreover, curl leaves the space $\hat{H}(\text{curl}, \Omega)^\perp$ invariant.

Proof. Note that by density of $\mathring{C}^\infty(\Omega)$ in $\hat{H}(\text{curl}, \Omega)$ both spaces have the same orthogonal complement. Hence,

$$\begin{aligned} f \in \hat{H}(\text{curl}, \Omega)^\perp &\Leftrightarrow 0 = \langle f, g \rangle_{L^2(\Omega)} + \langle \text{curl } f, \text{curl } g \rangle_{L^2(\Omega)} \quad \forall g \in \mathring{C}^\infty(\Omega) \\ &\Leftrightarrow \text{curl } f \in \hat{H}(\text{curl}, \Omega) \quad \text{and} \quad \text{curl } \text{curl } f = -f. \end{aligned} \quad \square$$

Lemma 4.11. *Let P be the orthogonal projection on $\hat{H}(\text{curl}, \Omega)^\perp$ (in $\hat{H}(\text{curl}, \Omega)$). Then $\hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega)$ is invariant under P , i.e., $f \in \hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega)$ implies $Pf \in \hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega)$.*

Proof. Since $I - P$ is the orthogonal projection on $\hat{H}(\text{curl}, \Omega)$ and $\hat{H}(\text{curl}, \Omega)$ is a subspace of $\hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega)$, we conclude that $(I - P)f \in \hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega)$ for every $f \in \hat{H}(\text{curl}, \Omega)$. Now for every $f \in \hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega)$ we have

$$Pf = f - (I - P)f,$$

which is in $\hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega)$, as linear combination of $\hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega)$ elements. \square

Lemma 4.12. *For every $q \in \pi_\tau(\hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega))$ there exists a $g \in \hat{H}(\text{curl}, \Omega)^\perp$ such that*

$$\text{curl } g \in \hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega) \cap \hat{H}(\text{curl}, \Omega)^\perp \quad \text{and} \quad \pi_\tau \text{curl } g = q.$$

In particular,

$$\pi_\tau(\hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega)) = \pi_\tau(\hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega) \cap \mathring{H}(\text{curl}, \Omega)^\perp).$$

Proof. By assumption we have $q = \pi_\tau f$ for $f \in \hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega)$. Let P denote the orthogonal projection on $\mathring{H}(\text{curl}, \Omega)^\perp$. Then by Lemma 4.11 we can decompose f into $f = Pf + (I - P)f$, where both Pf and $(I - P)f$ are also in $\hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega)$. Moreover, $(I - P)f \in \mathring{H}(\text{curl}, \Omega)$, which gives $\pi_\tau(I - P)f = 0$ and therefore

$$q = \pi_\tau f = \pi_\tau Pf.$$

Since $Pf \in \mathring{H}(\text{curl}, \Omega)^\perp$, we have $\text{curl curl } Pf = -Pf$. Thus defining $g = -\text{curl } Pf$ finishes the proof. \square

Now we finally come to our second main theorem.

Theorem 4.13. $\mathring{C}^\infty(\mathbb{R}^3)$ is dense in $\hat{H}_{\Gamma_1}(\text{curl}, \Omega)$ w.r.t. $\|\cdot\|_{\hat{H}_{\Gamma_1}(\text{curl}, \Omega)}$.

Proof. By the definition of the norm of $\hat{H}_{\Gamma_1}(\text{curl}, \Omega)$ the mapping $\gamma_\tau: \hat{H}_{\Gamma_1}(\text{curl}, \Omega) \subseteq H(\text{curl}, \Omega) \rightarrow L^2_\tau(\Gamma_1)$ is closed. We define the following restriction of γ_τ

$$\tilde{\gamma}_\tau: \begin{cases} \mathring{C}^\infty(\mathbb{R}^3) \subseteq H(\text{curl}, \Omega) & \rightarrow L^2_\tau(\Gamma_1), \\ f & \mapsto \gamma_\tau f. \end{cases}$$

Since $\tilde{\gamma}_\tau \subseteq \gamma_\tau$ we conclude

$$\tilde{\gamma}_\tau^* \supseteq \gamma_\tau^*.$$

1. Step: Calculate $\text{dom } \tilde{\gamma}_\tau^*$. Let $q \in \text{dom } \tilde{\gamma}_\tau^*$. Then there exists a $g \in H(\text{curl}, \Omega)$ such that

$$\langle \tilde{\gamma}_\tau f, q \rangle_{L^2(\Gamma_1)} = \langle f, g \rangle_{H(\text{curl}, \Omega)} = \langle f, g \rangle_{L^2(\Omega)} + \langle \text{curl } f, \text{curl } g \rangle_{L^2(\Omega)} \quad (2)$$

for all $f \in \mathring{C}^\infty(\mathbb{R}^3)$. For $f \in \mathring{C}_{\Gamma_1}^\infty(\mathbb{R}^3)$, we obtain

$$0 = \langle f, g \rangle_{L^2(\Omega)} + \langle \text{curl } f, \text{curl } g \rangle_{L^2(\Omega)},$$

which implies $\text{curl } g \in \mathring{H}_{\Gamma_0}(\text{curl}, \Omega)$ and $\text{curl curl } g = -g$, and by Lemma 4.10 $g \in \mathring{H}(\text{curl}, \Omega)^\perp$. Hence, we revisit (2), where we extend q by 0 outside of Γ_1 in $\partial\Omega$

$$\langle \tilde{\gamma}_\tau f, q \rangle_{L^2(\partial\Omega)} = -\langle f, \text{curl curl } g \rangle_{L^2(\Omega)} + \langle \text{curl } f, \text{curl } g \rangle_{L^2(\Omega)}$$

for all $f \in \mathring{C}^\infty(\mathbb{R}^3)$, which implies $\text{curl } g \in \hat{H}(\text{curl}, \Omega)$ and $q = \pi_\tau \text{curl } g$. Consequently,

$$\begin{aligned} \text{dom } \tilde{\gamma}_\tau^* &\subseteq \pi_\tau(\hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega) \cap \mathring{H}(\text{curl}, \Omega)^\perp) \\ &= \pi_\tau(\hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega)). \end{aligned}$$

2. Step: Calculate $\text{dom } \gamma_\tau^*$. If $q \in \pi_\tau(\hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega))$, then by Lemma 4.12 there exists a $g \in \mathring{H}(\text{curl}, \Omega)^\perp$ such that $\text{curl } g \in \hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega)$ and $\pi_\tau \text{curl } g = q$. Hence, by Corollary 4.9 for $f \in \hat{H}_{\Gamma_1}(\text{curl}, \Omega)$ and $\text{curl } g$ we have

$$\langle \gamma_\tau f, \underbrace{\gamma_\tau \text{curl } g}_{=q} \rangle_{L^2(\Gamma_1)} = \langle \text{curl } f, \text{curl } g \rangle_{L^2(\Omega)} - \langle f, \underbrace{\text{curl curl } g}_{=-g} \rangle_{L^2(\Omega)} = \langle f, g \rangle_{H(\text{curl}, \Omega)},$$

which implies $q \in \text{dom } \gamma_\tau^*$. Consequently,

$$\text{dom } \gamma_\tau^* \supseteq \pi_\tau(\hat{H}(\text{curl}, \Omega) \cap \mathring{H}_{\Gamma_0}(\text{curl}, \Omega)).$$

3. Step: Combining the results of the previous steps yields

$$\begin{aligned} \pi_\tau(\hat{H}(\text{curl}, \Omega) \cap \hat{H}_{\Gamma_0}(\text{curl}, \Omega)) &\supseteq \text{dom } \tilde{\gamma}_\tau^* \\ &\supseteq \text{dom } \gamma_\tau^* \supseteq \pi_\tau(\hat{H}(\text{curl}, \Omega) \cap \hat{H}_{\Gamma_0}(\text{curl}, \Omega)). \end{aligned}$$

Hence, $\tilde{\gamma}_\tau^* = \gamma_\tau^*$ and therefore

$$\overline{\tilde{\gamma}_\tau} = \tilde{\gamma}_\tau^{**} = \gamma_\tau^{**} = \gamma_\tau,$$

which implies $\dot{C}^\infty(\mathbb{R}^3)$ is dense in $\hat{H}_{\Gamma_1}(\text{curl}, \Omega)$ w.r.t. the graph norm of γ_τ which is $\|\cdot\|_{\hat{H}_{\Gamma_1}(\text{curl}, \Omega)}$. \square

4.3. Integration by parts. Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain and $\Gamma_0, \Gamma_1 \subseteq \partial\Omega$ a decomposition of $\partial\Omega$ such that (Ω, Γ_0) is a Lipschitz pair. The integration by parts formula

$$\langle \text{curl } f, g \rangle_{L^2(\Omega)} - \langle f, \text{curl } g \rangle_{L^2(\Omega)} = \langle \gamma_\tau f, \pi_\tau g \rangle_{L^2(\Gamma_1)}$$

is a priori valid for $f \in \dot{C}^\infty(\mathbb{R}^3)$ and $g \in \dot{C}_{\Gamma_0}^\infty(\mathbb{R}^3)$. Moreover, by definition of $\hat{H}_{\Gamma_1}(\text{curl}, \Omega)$ and $\hat{H}_{\partial\Omega}(\text{curl}, \Omega) \cap \hat{H}_{\Gamma_0}(\text{curl}, \Omega)$ we can extend this to either $f \in \hat{H}_{\Gamma_1}(\text{curl}, \Omega)$ or $g \in \hat{H}_{\partial\Omega}(\text{curl}, \Omega) \cap \hat{H}_{\Gamma_0}(\text{curl}, \Omega)$, but a priori not to both simultaneously.

As we have seen in Corollary 4.9 one density result is already sufficient to show that a simultaneous extension is valid. However, with both density results we conclude that this extension is even a continuous extension.

5. THE UNBOUNDED CASE

In this section we consider an unbounded Lipschitz domain $\Omega \subseteq \mathbb{R}^3$ such that (Ω, Γ_0) is a Lipschitz pair for $\Gamma_0 \subseteq \partial\Omega$ and we define $\Gamma_1 := \partial\Omega \setminus \overline{\Gamma_0}$.

The main idea is to multiply a vector field $f \in \hat{H}_{\Gamma_1}(\text{curl}, \Omega)$ with a cut-off function χ that is supported on a sufficiently large ball such that $\|f - \chi f\|_{\hat{H}_{\Gamma_1}(\text{curl}, \Omega)} \leq \epsilon$ for given $\epsilon > 0$. Then we only have approximate χf with smooth vector fields. This reduces the problem again to bounded domains, as the support of χf is bounded.

In order to execute these ideas we show some technical lemmas, that explain how the boundary conditions are inherited in a smooth cut-off process. Intuitively the lemmas are obvious, however we present them for completeness.

Lemma 5.1. *Let $f \in \hat{H}_{\Gamma_1}(\text{curl}, \Omega)$ and $\chi \in \dot{C}^\infty(\mathbb{R}^3)$. Then $\chi f \in \hat{H}_{\Gamma_1}(\text{curl}, \Omega)$ with $\pi_\tau(\chi f) = \chi \pi_\tau f$ on Γ_1 .*

Proof. Let $\phi \in \dot{C}_{\Gamma_0}^\infty(\mathbb{R}^3)$. Note that by the product rule for curl we have $\text{curl}(\chi f) = \chi \text{curl } f + \nabla \chi \times f$. Moreover, we have $\chi \phi \in \dot{C}_{\Gamma_0}^\infty(\mathbb{R}^3)$ and we denote $\pi_\tau f$ as $q \in L_\tau^2(\Gamma_1)$ on Γ_1 . Hence,

$$\begin{aligned} \langle \phi, \text{curl}(\chi f) \rangle_{L^2(\Omega)} &= \langle \phi, \chi \text{curl } f \rangle_{L^2(\Omega)} + \langle \phi, \nabla \chi \times f \rangle_{L^2(\Omega)} = \langle \chi \phi, \text{curl } f \rangle_{L^2(\Omega)} - \langle \nabla \chi \times \phi, f \rangle_{L^2(\Omega)} \\ &= \langle \text{curl}(\chi \phi), f \rangle_{L^2(\Omega)} + \langle \gamma_\tau(\chi \phi), q \rangle_{L_\tau^2(\Gamma_1)} - \langle \nabla \chi \times \phi, f \rangle_{L^2(\Omega)} \\ &= \langle \chi \text{curl } \phi, f \rangle_{L^2(\Omega)} + \langle \gamma_\tau \phi, \chi q \rangle_{L_\tau^2(\Gamma_1)}, \end{aligned}$$

which implies $\chi f \in \hat{H}_{\Gamma_1}(\text{curl}, \Omega)$ with $\gamma_\tau \chi f = \chi q$. \square

Corollary 5.2. *Let $f \in \hat{H}_{\partial\Omega}(\text{curl}, \Omega) \cap \hat{H}_{\Gamma_0}(\text{curl}, \Omega)$ and $\chi \in \dot{C}^\infty(\mathbb{R}^3)$. Then $\chi f \in \hat{H}_{\partial\Omega}(\text{curl}, \Omega) \cap \hat{H}_{\Gamma_0}(\text{curl}, \Omega)$ with $\pi_\tau(\chi f) = \chi \pi_\tau f$ on Γ_1 .*

Proof. By Lemma 5.1 applied with $\Gamma_1 = \partial\Omega$ we have $\chi f \in \hat{H}_{\partial\Omega}(\text{curl}, \Omega)$ and $\pi_\tau \chi f = \chi \pi_\tau f$. Since $\pi_\tau f = 0$ on Γ_0 we conclude that also $\pi_\tau \chi f = 0$ on Γ_0 and consequently $\chi f \in \hat{H}_{\Gamma_0}(\text{curl}, \Omega)$. \square

Lemma 5.3. *Let $f \in \hat{H}(\text{curl}, \Omega) \cap \dot{H}_{\Gamma_0}(\text{curl}, \Omega)$, $\alpha \in \dot{C}^\infty(C_{\epsilon,h}(p))$, $\Omega_p = \Omega \cap C_{\epsilon,h}(p)$ and $\Gamma_{1,p} = \Gamma_1 \cap \text{supp } \alpha$. Then $\alpha f \in \hat{H}(\text{curl}, \Omega_p) \cap \dot{H}_{\partial\Omega_i \setminus \overline{\Gamma_{1,p}}}(\text{curl}, \Omega_p)$ with $\pi_\tau \alpha f = \alpha \pi_\tau f$ on $\partial\Omega_p$, where $\pi_\tau f$ is extended by 0 where it is not defined.*

Proof. Note that $\alpha f \in \hat{H}(\text{curl}, \Omega) \cap \dot{H}_{\Gamma_0}(\text{curl}, \Omega)$ by Lemma 5.1. Hence, for $\phi \in \dot{C}^\infty(\mathbb{R}^3)$ we have

$$\begin{aligned} & \langle \text{curl } \phi, \alpha f \rangle_{L^2(\Omega_p)} - \langle \phi, \text{curl } \alpha f \rangle_{L^2(\Omega_p)} \\ &= \langle \text{curl } \phi, \alpha f \rangle_{L^2(\Omega)} - \langle \phi, \text{curl } \alpha f \rangle_{L^2(\Omega)} = \langle \gamma_\tau \phi, \alpha \pi_\tau f \rangle_{L^2(\Gamma_1)} \\ &= \langle \gamma_\tau \phi, \alpha \pi_\tau f \rangle_{L^2(\Gamma_{1,p})} = \langle \gamma_\tau \phi, \alpha \pi_\tau f \rangle_{L^2(\partial\Omega_p)} \end{aligned}$$

Hence, $\alpha f \in \hat{H}(\text{curl}, \Omega_p)$. Moreover, if we choose $\phi \in \dot{C}_{\Gamma_{1,p}}^\infty(\mathbb{R}^3)$ in the previous equation we conclude $\alpha f \in \dot{H}_{\partial\Omega \setminus \overline{\Gamma_{1,p}}}(\text{curl}, \Omega_p)$. \square

Remark 5.4. Note that in general the intersection of Lipschitz domains is not a Lipschitz domain. Roughly speaking this is because of the angle between the sets. However, for $p \in \partial\Omega$ the intersection $\Omega_p := C_{\epsilon,h}(p) \cap \Omega$ of the cylinder $C_{\epsilon,h}(p)$ and Ω is again a Lipschitz domain. Moreover, $(\Omega_p, \Gamma_0 \cap \Omega_p)$ is a Lipschitz pair. Roughly speaking here the assumptions on $C_{\epsilon,h}(p)$ make sure that angle is not 0.

Finally, we show that density result for unbounded domains.

Theorem 5.5. *Let Ω be an (unbounded) Lipschitz domain such that (Ω, Γ_0) is a Lipschitz pair. Then $\dot{C}_{\Gamma_0}^\infty(\mathbb{R}^3)$ is dense in $\hat{H}_{\partial\Omega}(\text{curl}, \Omega) \cap \dot{H}_{\Gamma_0}(\text{curl}, \Omega)$ w.r.t. $\|\cdot\|_{\hat{H}_{\partial\Omega}(\text{curl}, \Omega)}$.*

Proof. Let $f \in \hat{H}_{\partial\Omega}(\text{curl}, \Omega) \cap \dot{H}_{\Gamma_0}(\text{curl}, \Omega)$.

1. Step: Reduce to bounded case. Then for given ϵ there exists an $r > 0$ such that

$$\begin{aligned} \|f - \mathbb{1}_{B_r(0)} f\|_{L^2(\Omega)} &\leq \epsilon, \\ \|\text{curl } f - \mathbb{1}_{B_r(0)} \text{curl } f\|_{L^2(\Omega)} &\leq \epsilon, \\ \|\pi_\tau f - \mathbb{1}_{B_r(0)} \pi_\tau f\|_{L^2(\Omega)} &\leq \epsilon. \end{aligned}$$

This is a consequence of monotone convergence. We choose $\chi \in \dot{C}^\infty(\mathbb{R}^3)$ such that

$$\text{supp } \chi \subseteq B_{r+2}(0), \quad \chi(\zeta) \in [0, 1], \quad \chi = 1 \text{ on } B_r(0) \quad \text{and} \quad \|\nabla \chi\|_\infty \leq 1.$$

Such a function can be constructed by convolving $\mathbb{1}_{B_r(0)}$ with a suitable mollifier. By Lemma 5.1 we have $\chi f \in \hat{H}_{\partial\Omega}(\text{curl}, \Omega) \cap \dot{H}_{\Gamma_0}(\text{curl}, \Omega)$ with $\pi_\tau \chi f = \chi \pi_\tau f$. Moreover,

$$\begin{aligned} \|f - \chi f\|_{L^2(\Omega)} &\leq 2\epsilon, \\ \|\text{curl } f - \text{curl } \chi f\|_{L^2(\Omega)} &\leq \|\text{curl } f - \chi \text{curl } f\|_{L^2(\Omega)} + \|\nabla \chi \times f\|_{L^2(\Omega)} \leq 4\epsilon, \\ \|\pi_\tau f - \chi \pi_\tau f\|_{L^2(\Omega)} &\leq 2\epsilon. \end{aligned}$$

Hence, $\|f - \chi f\|_{\hat{H}_{\partial\Omega}(\text{curl}, \Omega)} \leq 8\epsilon$.

2. Step: Localize by partition of unity. We define $U_p = C_{\epsilon,h}(p)$ for $p \in \partial\Omega \cap B_{r+2}(0)$, where $C_{\epsilon,h}(p)$ is a cylinder that fits the Lipschitz assumptions of Ω and $U_p = B_\epsilon(p)$ for $p \in \Omega \cap B_{r+2}(0)$, where $\epsilon > 0$ is such that $B_\epsilon(p) \subseteq \Omega$. Then $(U_p)_{p \in (\partial\Omega \cap \Omega) \cap B_{r+2}(0)}$ is a covering of $\overline{\Omega \cap B_{r+2}(0)}$. Since $\overline{\Omega \cap B_{r+2}(0)}$ is compact there exists a finite subcovering $(U_i)_{i=1}^k$. We define $\Omega_i = U_i \cap \Omega$. Moreover, we employ a partition of unity subordinate to this covering and obtain functions $(\alpha_i)_{i=1}^k$ such that

$$\alpha_i \in \dot{C}^\infty(U_i), \quad \alpha_i(\zeta) \in [0, 1], \quad \text{and} \quad \sum_{i=1}^k \alpha_i(\zeta) = 1 \quad \text{for} \quad \zeta \in \overline{\Omega \cap B_{r+2}(0)}$$

We define $\Gamma_{1,i} := \Gamma_1 \cap \text{supp } \alpha_i$. Hence, if U_i is a cylinder, then $\alpha_i \chi f \in \hat{H}_{\partial\Omega_i}(\text{curl}, \Omega_i) \cap \dot{H}_{\partial\Omega_i \setminus \overline{\Gamma_{1,i}}}(\text{curl}, \Omega_i)$ and if U_i is a ball, then $\alpha_i \chi f \in \dot{H}_{\partial\Omega_i}(\text{curl}, \Omega_i)$. In both cases we

can extend these functions by 0 outside of Ω_i and obtain an element of $\hat{H}_{\partial\Omega}(\text{curl}, \Omega) \cap \hat{H}_{\Gamma_0}(\text{curl}, \Omega)$.

Moreover, we can approximate these functions by $\dot{C}_{\partial\Omega_i \setminus \Gamma_{1,i}}^\infty(\mathbb{R}^3)$ functions (that vanish on $\Omega \setminus \Omega_i$) w.r.t. $\|\cdot\|_{\hat{H}(\text{curl}, \Omega_i)}$ by Theorem 4.8. Since the extended versions are 0 outside of Ω_i we conclude this approximation also in $\hat{H}_{\partial\Omega}(\text{curl}, \Omega)$. Hence, we can approximate every $\alpha_i \chi f$ by a sequence in $\dot{C}_{\Gamma_0}^\infty(\mathbb{R}^3)$ (w.r.t. $\|\cdot\|_{\hat{H}_{\partial\Omega}(\text{curl}, \Omega)}$).

We denote these sequences by $(f_{i,n})_{n \in \mathbb{N}}$ and we choose $n \in \mathbb{N}$ so large that $\|\alpha_i \chi f - f_{i,n}\|_{\hat{H}_{\partial\Omega}(\text{curl}, \Omega)} \leq \frac{\epsilon}{k}$. Since $\chi f = 0$ outside of $B_{r+2}(0)$, we have $\sum_{i=1}^k \alpha_i \chi f = \chi f$ and therefore

$$\begin{aligned} \left\| f - \sum_{i=1}^k f_{i,n} \right\|_{\hat{H}_{\partial\Omega}(\text{curl}, \Omega)} &\leq \|f - \chi f\|_{\hat{H}_{\partial\Omega}(\text{curl}, \Omega)} + \sum_{i=1}^k \|\alpha_i \chi f - f_{i,n}\|_{\hat{H}_{\partial\Omega}(\text{curl}, \Omega)} \\ &\leq 8\epsilon + \epsilon. \end{aligned} \quad \square$$

Corollary 5.6. *Let Ω be an (unbounded) Lipschitz domain. Then $\dot{C}^\infty(\Omega)$ is dense in $\hat{H}(\text{curl}, \Omega)$ w.r.t. $\|\cdot\|_{\hat{H}(\text{curl}, \Omega)}$.*

Proof. This is just a special case of Theorem 5.5, in particular $\Gamma_0 = \partial\Omega$. \square

In order to generalize the other main result for unbounded domains we have two options: Either repeating the strategy of this section with small adaptations or repeating Section 4.2. The second option is more convenient as we only need to observe that boundedness was used for Theorem 4.8, which we can now replace by Theorem 5.5. Hence, we obtain the following theorem.

Theorem 5.7. *Let Ω be an (unbounded) Lipschitz domain such that (Ω, Γ_1) is a Lipschitz pair. Then $\dot{C}^\infty(\mathbb{R}^3)$ is dense in $\hat{H}_{\Gamma_1}(\text{curl}, \Omega)$ w.r.t. $\|\cdot\|_{\hat{H}_{\Gamma_1}(\text{curl}, \Omega)}$.*

6. CONCLUSION

We have defined $H(\text{curl}, \Omega)$ fields that possess an L^2 tangential trace on $\Gamma_1 \subseteq \partial\Omega$ via a weak approach (by representation in the $L^2(\Gamma_1)$ inner product) and showed that the C^∞ fields are dense in this space. This is a generalization of [BBBCD97], where the case $\Gamma_1 = \partial\Omega$ was regarded. Moreover, we lifted the result to unbounded domains. In fact for partial tangential traces there is the second question about the density with additional homogeneous boundary conditions on $\Gamma_0 = \partial\Omega \setminus \Gamma_1$. This was exactly the open problem in [WS13, Sec. 5], which we could solve. In particular they were asking whether $H_{\Gamma_0}^1(\Omega)$ is dense in $\hat{H}(\text{curl}, \Omega) \cap \hat{H}_{\Gamma_0}(\text{curl}, \Omega)$, which is in fact a weaker version of Theorem 4.8.

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