WEAK EQUALS STRONG L^2 REGULARITY FOR PARTIAL TANGENTIAL TRACES ON LIPSCHITZ DOMAINS

NATHANAEL SKREPEK AND DIRK PAULY

Abstract. We investigate the boundary trace operators that naturally correspond to $\mathrm{H}(\mathrm{curl}, \Omega)$, namely the tangential and twisted tangential trace, where $\Omega \subseteq \mathbb{R}^3$. In particular we regard partial tangential traces, i.e., we look only on a subset Γ of the boundary $\partial\Omega$. We assume both Ω and Γ to be strongly Lipschitz. We define the space of all $\mathrm{H}(\mathrm{curl}, \Omega)$ fields that possess a L^2 tangential trace in a weak sense and show that the set of all smooth fields is dense in that space, which is a generalization of [BBBCD97]. This is especially important for Maxwell's equation with mixed boundary condition as we answer the open problem by Weiss and Staffans in [WS13, Sec. 5] for strongly Lipschitz pairs.

1. Introduction

We will regard a bounded strongly Lipschitz domain $\Omega \subseteq \mathbb{R}^3$ and the Sobolev space that corresponds to the curl operator

$$\mathrm{H}(\mathrm{curl}, \Omega) = \{f \in \mathrm{L}^2(\Omega) \mid \mathrm{curl}\, f \in \mathrm{L}^2(\Omega)\}$$

and the "natural" boundary traces that are associated with the curl operator

$$\pi_\tau f := \nu \times f\big|_{\partial\Omega} \times \nu \quad \text{and} \quad \gamma_\tau f := \nu \times f\big|_{\partial\Omega} \quad \text{for} \quad f \in \mathrm{C}^\infty(\mathbb{R}^3),$$

where ν denotes the outer normal vector on the boundary of Ω. These boundary traces are called *tangential trace* and *twisted tangential trace*, respectively. They are motivated by the integration by parts formula

$$\langle \mathrm{curl}\, f, g\rangle_{\mathrm{L}^2(\Omega)} - \langle f, \mathrm{curl}\, g\rangle_{\mathrm{L}^2(\Omega)} = \langle \gamma_\tau f, \pi_\tau g\rangle_{\mathrm{L}^2(\partial\Omega)}.$$

We can even extend these boundary operators to $\mathrm{H}(\mathrm{curl}, \Omega)$ by introducing suitable boundary spaces, see e.g., [BCS02] for full boundary traces or [Skr21] for partial boundary traces. However, in this article we focus on those $f \in \mathrm{H}(\mathrm{curl}, \Omega)$ that have a meaningful $\mathrm{L}^2(\partial\Omega)$ (twisted) tangential trace. Hence, for $\Gamma \subseteq \partial\Omega$ we are interested in the following spaces

$$\mathring{\mathrm{H}}_\Gamma(\mathrm{curl}, \Omega) = \{f \in \mathrm{H}(\mathrm{curl}, \Omega) \mid \pi_\tau f = 0 \text{ on } \Gamma\},$$
$$\hat{\mathrm{H}}_\Gamma(\mathrm{curl}, \Omega) = \{f \in \mathrm{H}(\mathrm{curl}, \Omega) \mid \pi_\tau f \text{ is in } \mathrm{L}^2(\Gamma)\}.$$

where we will later state precisely what we mean by $\pi_\tau f = 0$ on Γ and $\pi_\tau f \in \mathrm{L}^2(\Gamma)$. In particular we are interested in $\hat{\mathrm{H}}_\Gamma(\mathrm{curl}, \Omega)$. Similar to Sobolev spaces there are two approaches to $\pi_\tau f \in \mathrm{L}^2(\Gamma)$: A weak approach by representation in an inner product and a strong approach by limits of regular functions. We use the weak approach as definition, see Definition 4.1. The question that immediately arises is: *"Do both approaches lead to the same space?"*

In [WS13, eq. (5.20)] the authors observed this problem and concluded that it can cause ambiguity for boundary conditions, if the approaches don't coincide. In

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E-mail: nathanael.skrepek@math.tu-freiberg.de.

fact they stated this issue at the end of section 5 in [WS13] as an open problem. This problem can actually be viewed as a more general question that arises for quasi Gelfand triples, see [Skr23b, Conjecture 6.7].

We will not explicitly define the strong approach, but show that the most regular functions (C^∞ functions) are already dense in the weakly defined space, which immediately implies that any strong approach with less regular functions (e.g., H^1) will lead to the same space. This is exactly what was done in [BBBCD97] for $\Gamma = \partial\Omega$. Hence, we present a generalization of [BBBCD97] for partial L^2 tangential traces. In particular, we aim to prove the following two main theorems.

Theorem 1.1. *Let Ω be a bounded strongly Lipschitz domain and $\Gamma_1 \subseteq \partial\Omega$ such that (Ω, Γ_1) is a strongly Lipschitz pair, then $\mathring{\mathrm{C}}^\infty(\mathbb{R}^3)$ is dense in $\hat{\mathrm{H}}_{\Gamma_1}(\mathrm{curl}, \Omega)$ with respect to $\|\cdot\|_{\hat{\mathrm{H}}_{\Gamma_1}(\mathrm{curl},\Omega)}$.*

Theorem 1.2. *Let Ω be a bounded strongly Lipschitz domain and $\Gamma_0 \subseteq \partial\Omega$ such that (Ω, Γ_0) is a strongly Lipschitz pair, then $\mathring{\mathrm{C}}^\infty_{\Gamma_0}(\mathbb{R}^3)$ is dense in $\hat{\mathrm{H}}_{\partial\Omega}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$ with respect to $\|\cdot\|_{\hat{\mathrm{H}}_{\partial\Omega}(\mathrm{curl},\Omega)}$.*

However, it turned out that it is best to prove them in reversed order.

The importance of our density results lies in the context of Maxwell's equations with boundary conditions that involve a mixture of π_τ and γ_τ in the sense of linear combination, e.g., this simplified instance of Maxwell's equations

$$\begin{aligned} \partial_t E(t,\zeta) &= \mathrm{curl}\, H(t,\zeta), && t \geq 0, \zeta \in \Omega, \\ \partial_t H(t,\zeta) &= -\,\mathrm{curl}\, E(t,\zeta), && t \geq 0, \zeta \in \Omega, \\ \pi_\tau E(t,\xi) + \gamma_\tau H(t,\xi) &= 0, && t \geq 0, \xi \in \Gamma_1, \\ \pi_\tau E(t,\xi) &= 0, && t \geq 0, \xi \in \Gamma_0. \end{aligned}$$

In order to properly formulate the boundary conditions we need to know what functions E, H have tangential traces that allow such a linear combination. Especially when it comes to well-posedness our density results are needed to avoid the ambiguity that was observed in [WS13].

As suspected by Weiss and Staffans in [WS13] the regularity of the interface of $\Gamma_0 \subseteq \partial\Omega$ and $\Gamma_1 \coloneqq \partial\Omega \setminus \overline{\Gamma_0}$ seems to play a role. At least for our answer we need that the boundary of Γ_0 is also strongly Lipschitz.

In particular our strategy is based on the following decomposition from [PS22a, Thm. 5.2]

$$\mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega) = \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega) + \nabla \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega), \tag{1}$$

which requires (Ω, Γ_0) to be a strongly Lipschitz pair. Every element of $\mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$ can be approximated by a sequence in $\mathring{\mathrm{C}}^\infty_{\Gamma_0}(\mathbb{R}^3)$ w.r.t. $\|\cdot\|_{\mathrm{H}^1(\Omega)}$ (see [BPS16, Lmm. 3.1]), which is a stronger norm than the "natural" norm of $\hat{\mathrm{H}}_{\partial\Omega}(\mathrm{curl}, \Omega)$. Hence, the challenging part will be finding an approximation by $\mathring{\mathrm{C}}^\infty_{\Gamma_0}(\mathbb{R}^3)$ elements for all elements in

$$\hat{\mathrm{H}}_{\partial\Omega}(\mathrm{curl}, \Omega) \cap \nabla \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega).$$

It even turned out that, if we can prove the decomposition (1) also for less regular Γ_0, then our main theorems would automatically generalize for those less regular partitions of $\partial\Omega$, since this is the only occasion where the regularity of Γ_0 is used.

2. Preliminary

For $\Omega \subseteq \mathbb{R}^d$ open and $\Gamma \subseteq \partial\Omega$ open we use the following notation (as in [BPS16])

$$\mathring{C}^\infty(\Omega) := \{ f \in C^\infty(\Omega) \,|\, \operatorname{supp} f \text{ is compact in } \Omega \},$$
$$\mathring{C}^\infty_\Gamma(\Omega) := \Big\{ f\big|_\Omega \,\Big|\, f \in \mathring{C}^\infty(\mathbb{R}^d), \operatorname{dist}(\Gamma, \operatorname{supp} f) > 0 \Big\},$$

and $H^1(\Omega)$ denotes the usual Sobolev space and $\mathring{H}^1_\Gamma(\Omega)$ is the subspace of $H^1(\Omega)$ with homogeneous boundary data on Γ, i.e., $\mathring{H}^1_\Gamma(\Omega) = \overline{\mathring{C}^\infty_\Gamma(\Omega)}^{H^1(\Omega)}$.

Note that the trace operators π_τ and γ_τ are called tangential traces, because $\nu \cdot \pi_\tau f = 0$ and $\nu \cdot \gamma_\tau f = 0$. Hence, it is natural to introduce the *tangential* L^2 space on $\Gamma \subseteq \partial\Omega$ by

$$L^2_\tau(\Gamma) = \{ f \in L^2(\Gamma) \,|\, \nu \cdot f = 0 \}.$$

This space is again a Hilbert space with the $L^2(\Gamma)$ inner product. Moreover, both $\pi_\tau \mathring{C}^\infty_{\partial\Omega\setminus\Gamma}(\mathbb{R}^3)$ and $\gamma_\tau \mathring{C}^\infty_{\partial\Omega\setminus\Gamma}(\mathbb{R}^3)$ are dense in that space.

Next we recall the definition of a strongly Lipschitz domain, see e.g., [Gri85]. Moreover, we need H^1 spaces on strongly Lipschitz boundaries, see e.g, [Skr23a] for a careful treatment.

Definition 2.1. Let Ω be an open subset of $\mathbb{R}^d$. We say Ω is a *strongly Lipschitz domain*, if for every $p \in \partial\Omega$ there exist $\epsilon, h > 0$, a hyperplane $W = \operatorname{span}\{w_1, \dots, w_{d-1}\}$, where $\{w_1, \dots, w_{d-1}\}$ is an orthonormal basis of W, and a Lipschitz continuous function $a\colon (p + W) \cap B_\epsilon(p) \to (-\frac{h}{2}, \frac{h}{2})$ such that

$$\partial\Omega \cap C_{\epsilon,h}(p) = \{ x + a(x)v \,|\, x \in (p + W) \cap B_\epsilon(p) \},$$
$$\Omega \cap C_{\epsilon,h}(p) = \{ x + sv \,|\, x \in (p + W) \cap B_\epsilon(p), -h < s < a(x) \},$$

where v is the normal vector of W and $C_{\epsilon,h}(p)$ is the cylinder $\{x + \delta v \,|\, x \in (p + W) \cap B_\epsilon(p), \delta \in (-h, h)\}$.

The boundary $\partial\Omega$ is then called *strongly Lipschitz boundary.*

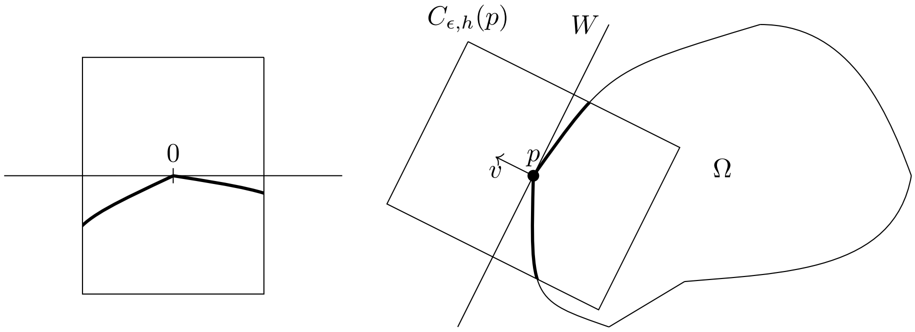


Figure 1. Lipschitz boundary

Corresponding to a strongly Lipschitz domain we define the following bi-Lipschitz continuous mapping

$$k\colon \left\{ \begin{array}{rcl} \partial\Omega \cap C_{\epsilon,h}(p) & \to & B_\epsilon(0) \subseteq \mathbb{R}^{d-1}, \\ \zeta & \mapsto & W^\mathsf{T}(\zeta - p), \end{array} \right.$$

where we used W as the matrix $[\, w_1 \cdots w_{d-1} \,]$. We call this mapping a *regular Lipschitz chart* of $\partial\Omega$ and we call its domain the *chart domain.* Its inverse is given by

$$k^{-1}\colon \begin{cases} \mathrm{B}_\epsilon(0) \subseteq \mathbb{R}^{d-1} & \to \quad \partial\Omega \cap C_{\epsilon,h}(p), \\ \qquad\qquad x & \mapsto \quad p + Wx + a(x)v, \end{cases}$$

where we will use $a(x)$ also as shortcut for $a(p+Wx)$, which is then a Lipschitz continuous function from $\mathrm{B}_\epsilon(0) \subseteq \mathbb{R}^{d-1}$ to $\mathbb{R}$. Charts are used to regard the surface of Ω locally as a flat subset of $\mathbb{R}^{d-1}$. Every restriction of a chart k to an open $\Gamma \subseteq \partial\Omega$ is again a chart. The shape of $k(\Gamma)$, which is the image of the restricted chart, can be less "regular" than the nice shape of the ball $\mathrm{B}_\epsilon(0)$, which was the original image. Hence, for some investigations such restricted charts are not suitable. Therefore, we call such a restricted chart in general just *Lipschitz chart* in contrast to regular Lipschitz charts.

Definition 2.2. Let Ω be a strongly Lipschitz domain in $\mathbb{R}^d$. Then we say that an open $\Gamma_0 \subseteq \partial\Omega$ is strongly Lipschitz, if $k(\Gamma_0)$ is strongly Lipschitz domain in $\mathbb{R}^{d-1}$ for all regular Lipschitz charts k of $\partial\Omega$.

The boundary $\partial\Gamma_0$ is then called *strongly Lipschitz boundary*.

Note that it is sufficient that the image of Γ_0 under k (in the previous definition) is strongly Lipschitz for a set of regular Lipschitz charts, whose chart domains cover Γ_0 (or even just $\partial\Gamma_0$).

Definition 2.3. We call (Ω, Γ_0) a *strongly Lipschitz pair*, if Ω is a strongly Lipschitz domain and $\Gamma_0 \subseteq \partial\Omega$ is strongly Lipschitz.

Note that if $\Gamma_0 \subseteq \partial\Omega$ is strongly Lipschitz, then also $\Gamma_1 \coloneqq \partial\Omega \setminus \overline{\Gamma_0}$ is strongly Lipschitz. Hence, if (Ω, Γ_0) is a strongly Lipschitz pair, then also (Ω, Γ_1) is.

Since we only deal with strongly Lipschitz domains and boundaries, we will omit the term "strongly" and just say *Lipschitz domain* and *Lipschitz boundary*.

Recall the definition of a H^1 function on the boundary of a Lipschitz domain, see e.g., [Skr23a].

Definition 2.4. Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz domain. We say $f \in \mathrm{L}^2(\partial\Omega)$ is in $\mathrm{H}^1(\partial\Omega)$, if for every Lipschitz chart $k\colon \Gamma \to U$ the mapping

$$f \circ k^{-1} \quad \text{is in} \quad \mathrm{H}^1(U).$$

3. Density results for $W(\Omega)$

Definition 3.1. Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz domain. Then we define

$$W(\Omega) \coloneqq \big\{ f \in \mathrm{H}^1(\Omega) \,\big|\, \gamma_0 f \in \mathrm{H}^1(\partial\Omega) \big\},$$

$$\|f\|_{W(\Omega)} \coloneqq \Big(\|f\|^2_{\mathrm{H}^1(\Omega)} + \|\gamma_0 f\|^2_{\mathrm{H}^1(\partial\Omega)} \Big)^{1/2}.$$

The next lemma a is a crucial tool in our construction. The basic idea is: Take a smooth function with compact support on a flat domain ($U \subseteq \mathbb{R}^{d-1}$) extend it on the entire hyperplane $\mathbb{R}^{d-1}$ by 0, and then extend is constantly in the orthogonal direction, i.e., $f(\zeta + \lambda e_d) = f(\zeta)$, where $\lambda \in \mathbb{R}$ and e_d is the d-th unit vector. A multiplication with a cutoff function makes sure that this extension has compact support. By rotation and translation this can be done for arbitrary hyperplanes. Figure 2 illustrates the construction.

Lemma 3.2. *Let $k\colon \Gamma \to U$ be a Lipschitz chart, $f \in \mathrm{H}^1(\partial\Omega)$ with compact support in $\Gamma' \subseteq \Gamma$. Then there exists an $F \in \mathrm{H}^1(\mathbb{R}^d) \cap W(\Omega) \cap \mathring{\mathrm{H}}^1_{\partial\Omega\setminus\Gamma'}(\Omega)$ such that $F\big|_{\partial\Omega} = f$. Moreover, there exists a sequence $(F_n)_{n\in\mathbb{N}}$ in $\mathring{\mathrm{C}}^\infty_{\partial\Omega\setminus\Gamma'}(\mathbb{R}^d)$ that converges to F w.r.t. $\|\cdot\|_{\mathrm{H}^1(\mathbb{R}^d)} + \|\cdot\|_{W(\Omega)}$, i.e., F_n converges to F in $\mathrm{H}^1(\mathbb{R}^d)$ and $F_n\big|_{\partial\Omega}$ converges to $F\big|_{\partial\Omega}$ in $\mathrm{H}^1(\partial\Omega)$.*

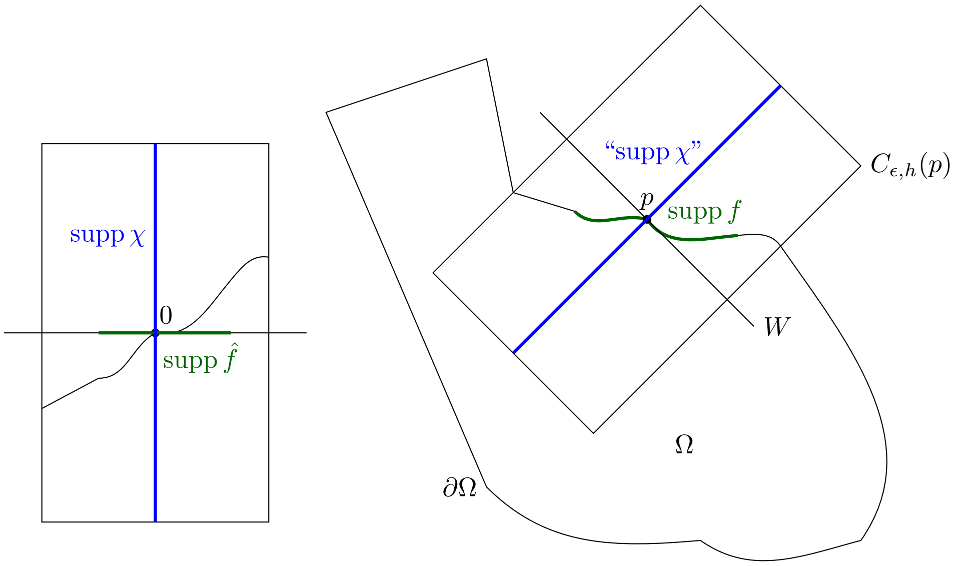


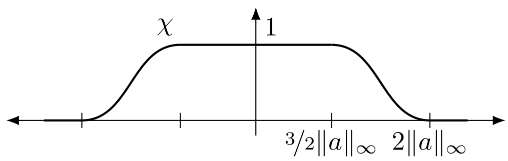
FIGURE 2. Illustration of the construction of Lemma 3.2

Proof. Let p, W and v be the point, hyperplane and normal vector, respectively, to the chart k. In particular k^{-1} is given by

$$k^{-1}\colon \begin{cases} U \subseteq \mathbb{R}^{d-1} & \to \quad \Gamma, \\ \qquad\quad x & \mapsto \quad p + Wx + a(x)v, \end{cases}$$

where U is open and a is the Lipschitz function. Let $\chi \in \mathring{C}^\infty(\mathbb{R})$ be a cut-off function such that

$$\chi(\lambda) \in \begin{cases} \{1\}, & |\lambda| < {}^3\!/\!_2 \|a\|_\infty, \\ [0,1], & |\lambda| \in ({}^3\!/\!_2, 2)\|a\|_\infty, \\ \{0\}, & |\lambda| > 2\|a\|_\infty. \end{cases}$$



By definition $\hat f = f \circ k^{-1}$ is in $H^1(U)$ and since f has compact support in Γ' we conclude $\hat f \in \mathring{H}^1(U)$ with support in $U' := k(\Gamma')$ Note that we can extend $\hat f \in \mathring{H}^1(U)$ on $\mathbb{R}^d$ by 0. We define

$$F(\zeta) = \chi(v \cdot (\zeta - p)) \hat f\big(W^\mathsf{T}(\zeta - p)\big) \quad \text{for} \quad \zeta \in \mathbb{R}^d$$

The support of F is inside of a rotated and translated version of $U' \times \operatorname{supp}\chi$, in particular

$$\operatorname{supp} F \subseteq p + \begin{bmatrix} W & v \end{bmatrix} U' \times \operatorname{supp}\chi =: \Xi.$$

Note that by construction of χ we have $\operatorname{supp} F \cap \partial\Omega \subseteq \Gamma'$, therefore $F\big|_{\partial\Omega\setminus\Gamma'} = 0$. Since $\hat f \in H^1(\mathbb{R}^{d-1})$ it is straight forward that F possess $L^2(\mathbb{R}^d)$ directional derivatives in W directions. Moreover, by construction (and the Leibniz product rule) $\frac{\partial}{\partial v} F = \chi' \hat f(W^\mathsf{T}(\cdot - p))$, which implies $F \in H^1(\mathbb{R}^d)$. By definition of a Lipschitz chart we have $|v \cdot (\zeta - p)| \leq \|a\|_\infty$ for $\zeta \in \Gamma$ and hence

$$F(\zeta) = \underbrace{\chi(v \cdot (\zeta - p))}_{=1} \hat f(W^\mathsf{T}(\zeta - p)) = \hat f \circ k(\zeta) = f(\zeta) \quad \text{for} \quad \zeta \in \Gamma$$

(a.e. w.r.t. the surface measure).

By assumption on $\hat{f}$ there exists a sequence $(\hat{f}_n)_{n\in\mathbb{N}}$ in $\mathring{\mathrm{C}}^\infty(U)$ that converges to $\hat{f}$ w.r.t. $\|\cdot\|_{\mathrm{H}^1(U)}$. Note that $\hat{f}_n$ is also in $\mathring{\mathrm{C}}^\infty(\mathbb{R}^{d-1})$. We define

$$F_n(\zeta) = \chi(v \cdot (\zeta - p))\hat{f}_n\big(W^\mathsf{T}(\zeta - p)\big) \quad \text{for} \quad \zeta \in \mathbb{R}^d$$

Note that F_n is the composition of C^∞ mappings and therefore in $\mathrm{C}^\infty(\mathbb{R}^d)$. Again, the support of F_n is contained in the bounded set Ξ and therefore compact, which implies $F_n \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^d)$. Note that $F_n \circ k^{-1} = \hat{f}_n$, which implies $(F_n \circ k^{-1})_{n\in\mathbb{N}}$ converges to $\hat{f}$ w.r.t. $\|\cdot\|_{\mathrm{H}^1(U)}$. Since $F_n\big|_{\partial\Omega\setminus\Gamma} = 0 = F\big|_{\partial\Omega\setminus\Gamma}$ we conclude $F_n\big|_{\partial\Omega} \to F\big|_{\partial\Omega}$ in $\mathrm{H}^1(\partial\Omega)$. Finally,

$$\begin{aligned}\|F_n - F\|_{\mathrm{H}^1(\mathbb{R}^3)} &\leq \|\chi'\|_\infty \|(\hat{f}_n - \hat{f})(W^\mathsf{T}(\cdot - p))\|_{\mathrm{H}^1(\Xi)} \\ &\leq 2\|a\|_\infty \|\chi'\|_\infty \|\hat{f}_n - \hat{f}\|_{\mathrm{H}^1(U)} \to 0. \qquad ❑\end{aligned}$$

We will formulate a generalization of [BBBCD97, 2. Preliminaries].

Theorem 3.3. *$\mathring{\mathrm{C}}^\infty_\Gamma(\mathbb{R}^d)$ is dense in $W(\Omega) \cap \mathring{\mathrm{H}}^1_\Gamma(\Omega)$ w.r.t. $\|\cdot\|_{W(\Omega)}$.*

Proof. Since Ω is a Lipschitz domain we find for every $p \in \partial\Omega$ a cylinder $C_{\epsilon,h}(p)$ (ϵ and h depend on p) and a Lipschitz chart $k\colon \partial\Omega \cap C_{\epsilon,h}(p) \to \mathrm{B}_\epsilon(0) \subseteq \mathbb{R}^{d-1}$.

Hence we can cover $\partial\Omega$ by $\bigcup_{p\in\partial\Omega} C_{\epsilon,h}(p)$ and by compactness of $\partial\Omega$ there are already finitely many p_i, $i \in \{1, \dots m\}$ such that

$$\partial\Omega \subseteq \bigcup_{i=1}^m \underbrace{C_{\epsilon_i,h_i}(p_i)}_{=:\Omega_i}$$

We employ a partition of unity and obtain $(\alpha_i)_{i=1}^m$, subordinate to this cover, i.e.,

$$\alpha_i \in \mathring{\mathrm{C}}^\infty(\Omega_i), \quad \alpha_i(\zeta) \in [0,1], \quad \text{and} \quad \sum_{i=1}^m \alpha_i(\zeta) = 1 \quad \text{for all} \quad \zeta \in \partial\Omega.$$

For $f \in W(\Omega) \cap \mathring{\mathrm{H}}^1_\Gamma(\Omega)$ we define $f_i \coloneqq \alpha_i f$. It is straightforward to show $f_i \in W(\Omega) \cap \mathring{\mathrm{H}}^1_\Gamma(\Omega)$. For every Ω_i there is a Lipschitz chart $k_i\colon \Gamma_i \to U_i \subseteq \mathbb{R}^{d-1}$, where $\Gamma_i = \partial\Omega \cap \Omega_i$. Moreover, $f_i\big|_{\partial\Omega}$ has support in $\Gamma_i \cap \Gamma^\complement$, where $\Gamma^\complement = (\partial\Omega \setminus \Gamma)$.

By Lemma 3.2 there is an $F_i \in \mathrm{H}^1(\mathbb{R}^d) \cap W(\Omega) \cap \mathring{\mathrm{H}}_{\partial\Omega\setminus(\Gamma_i\cap\Gamma^\complement)}$ such that $F_i\big|_{\partial\Omega} = f_i\big|_{\partial\Omega}$ and a sequence $(F_{i,n})_{n\in\mathbb{N}}$ in $\mathring{\mathrm{C}}^\infty_{\partial\Omega\setminus(\Gamma_i\cap\Gamma^\complement)}(\mathbb{R}^d) \subseteq \mathring{\mathrm{C}}^\infty_\Gamma(\mathbb{R}^d)$ that converges to F_i in $\mathrm{H}(\mathbb{R}^d)$ and in $W(\Omega)$. Hence, we have

$$f - \sum_{i=1}^m F_i \in \mathring{\mathrm{H}}^1(\Omega),$$

which can be approximated by $(F_{0,n})_{n\in\mathbb{N}}$ in $\mathring{\mathrm{C}}^\infty(\Omega)$. Hence, $\big(\sum_{i=0}^m F_{i,n}\big)_{n\in\mathbb{N}}$ is a sequence in $\mathring{\mathrm{C}}^\infty_\Gamma(\mathbb{R}^d)$ and converges to f in $W(\Omega)$. ❑

4. Density result with homogeneous part

In this section we will finally define the Sobolev spaces with homogeneous and L^2 partial tangential traces, respectively, and prove one of our main theorems. We assume $\Omega \subseteq \mathbb{R}^3$ to be a Lipschitz domain.

We will use a weak definition for the tangential trace and twisted tangential trace as, e.g., in [PS22b].

Definition 4.1. Let Ω be a Lipschitz domain and $\Gamma \subseteq \partial\Omega$ open (in $\partial\Omega$).

- We say $f \in \mathrm{H}(\mathrm{curl}, \Omega)$ has a $\mathrm{L}^2_\tau(\Gamma)$ tangential trace, if there exists a $q \in \mathrm{L}^2_\tau(\Gamma)$ such that
$$\langle f, \mathrm{curl}\,\phi\rangle_{\mathrm{L}^2(\Omega)} - \langle \mathrm{curl}\, f, \phi\rangle_{\mathrm{L}^2(\Omega)} = \langle q, \gamma_\tau \phi\rangle_{\mathrm{L}^2_\tau(\Gamma)} \quad \forall \phi \in \mathring{\mathrm{C}}^\infty_{\partial\Omega\setminus\Gamma}(\mathbb{R}^3).$$
In this case we say $\pi_\tau f \in \mathrm{L}^2_\tau(\Gamma)$ and $\pi_\tau f = q$ on Γ or more precisely $\pi^\Gamma_\tau f = q$.
- We say $f \in \mathrm{H}(\mathrm{curl}, \Omega)$ has a $\mathrm{L}^2_\tau(\Gamma)$ twisted tangential trace, if there exists a $q \in \mathrm{L}^2_\tau(\Gamma)$ such that
$$\langle \mathrm{curl}\, f, \phi\rangle_{\mathrm{L}^2(\Omega)} - \langle f, \mathrm{curl}\,\phi\rangle_{\mathrm{L}^2(\Omega)} = \langle q, \pi_\tau \phi\rangle_{\mathrm{L}^2_\tau(\Gamma)} \quad \forall \phi \in \mathring{\mathrm{C}}^\infty_{\partial\Omega\setminus\Gamma}(\mathbb{R}^3).$$
In this case we say $\gamma_\tau f \in \mathrm{L}^2_\tau(\Gamma)$ and $\gamma_\tau f = q$ on Γ or more precisely $\gamma^\Gamma_\tau f = q$.

Note that the previous definition does not say anything about $\pi_\tau f$ on $\partial\Omega \setminus \Gamma$.

Remark 4.2. Note that $\nu \times \gamma_\tau \phi = -\pi_\tau \phi$ and $\langle q, \gamma_\tau \phi\rangle_{\mathrm{L}^2_\tau(\Gamma)} = \langle \nu \times q, \nu \times \gamma_\tau \phi\rangle_{\mathrm{L}^2_\tau(\Gamma)}$. Hence, we can easily see that $\pi_\tau f \in \mathrm{L}^2(\Gamma)$ is equivalent to $\gamma_\tau f \in \mathrm{L}^2(\Gamma)$ and $\gamma_\tau f = \nu \times \pi_\tau f$.

Definition 4.3. Let Ω be a Lipschitz domain and $\Gamma \subseteq \partial\Omega$ open (in $\partial\Omega$). Then we define the space
$$\hat{\mathrm{H}}_\Gamma(\mathrm{curl}, \Omega) := \{f \in \mathrm{H}(\mathrm{curl}, \Omega) \,|\, \pi_\tau f \in \mathrm{L}^2_\tau(\Gamma)\}$$
with its norm
$$\|f\|_{\hat{\mathrm{H}}_\Gamma(\mathrm{curl},\Omega)} := \Big(\|f\|^2_{\mathrm{L}^2(\Omega)} + \|\mathrm{curl}\, f\|^2_{\mathrm{L}^2(\Omega)} + \|\pi_\tau f\|^2_{\mathrm{L}^2(\Gamma)}\Big)^{1/2}.$$
For $\Gamma = \partial\Omega$ we will just write $\hat{\mathrm{H}}(\mathrm{curl}, \Omega)$ instead of $\hat{\mathrm{H}}_{\partial\Omega}(\mathrm{curl}, \Omega)$.

Definition 4.4. Let Ω be a Lipschitz domain and $\Gamma \subseteq \partial\Omega$ open (in $\partial\Omega$). Then we define the space
$$\mathring{\mathrm{H}}_\Gamma(\mathrm{curl}, \Omega) = \{f \in \hat{\mathrm{H}}_\Gamma(\mathrm{curl}, \Omega) \,|\, \pi^\Gamma_\tau f = 0\}.$$
For $\Gamma = \partial\Omega$ we will just write $\mathring{\mathrm{H}}(\mathrm{curl}, \Omega)$ instead of $\mathring{\mathrm{H}}_{\partial\Omega}(\mathrm{curl}, \Omega)$.

In [BPS16, Thm. 4.5] it is shown that $\mathring{\mathrm{C}}^\infty_\Gamma(\Omega)$ is dense in $\mathring{\mathrm{H}}_\Gamma(\mathrm{curl}, \Omega)$ w.r.t. $\|\cdot\|_{\mathrm{H}(\mathrm{curl},\Omega)}$, i.e.,
$$\mathring{\mathrm{H}}_\Gamma(\mathrm{curl}, \Omega) = \overline{\mathring{\mathrm{C}}^\infty_\Gamma(\Omega)}^{\mathrm{H}(\mathrm{curl},\Omega)}$$
Hence, for homogeneous tangential traces there is already a version of the desired density result.

Note that the hat on top of the H indicates partial L^2 boundary conditions and the circle on top indicates partial homogeneous boundary conditions.

Remark 4.5. We can immediately see
$$\mathring{\mathrm{H}}_\Gamma(\mathrm{curl}, \Omega) \subseteq \hat{\mathrm{H}}_\Gamma(\mathrm{curl}, \Omega).$$
Since $\pi_\tau f \in \mathrm{L}^2(\Gamma)$ is equivalent to $\gamma_\tau f \in \mathrm{L}^2(\Gamma)$ we have
$$\hat{\mathrm{H}}_\Gamma(\mathrm{curl}, \Omega) = \{f \in \mathrm{H}(\mathrm{curl}, \Omega) \,|\, \gamma_\tau f \in \mathrm{L}^2(\Gamma)\},$$
Since $\pi_\tau f = \gamma_\tau f \times \nu$, we have $\|\pi_\tau f\|_{\mathrm{L}^2(\Gamma)} = \|\gamma_\tau f\|_{\mathrm{L}^2(\Gamma)}$ and
$$\|f\|_{\hat{\mathrm{H}}_\Gamma(\mathrm{curl},\Omega)} = \big(\|f\|^2_{\mathrm{L}^2(\Omega)} + \|\mathrm{curl}\, f\|^2_{\mathrm{L}^2(\Omega)} + \|\gamma_\tau f\|^2_{\mathrm{L}^2(\Gamma)}\big)^{1/2}.$$

Remark 4.6. Since we use representation in an inner product, one can say that we have defined $\hat{\mathrm{H}}_\Gamma(\mathrm{curl}, \Omega)$ weakly. Another approach could have been to regard $\overline{\mathring{\mathrm{C}}^\infty(\mathbb{R}^3)}^{\hat{\mathrm{H}}_\Gamma(\mathrm{curl},\Omega)}$, which could be called a strong approach. From this perspective the result we are going to show basically tells us that the weak and the strong approach to $\mathrm{H}(\mathrm{curl}, \Omega)$ fields that possess a $\mathrm{L}^2_\tau(\Gamma)$ tangential trace coincide.

From now on we assume that (Ω, Γ_0) is a Lipschitz pair. Recall the decomposition (1):

$$\mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega) = \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega) + \nabla \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega).$$

Note that every element in $\mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$ is already in $\hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$. Moreover, by [BPS16, Lmm. 3.1] $\mathring{\mathrm{C}}^\infty_{\Gamma_0}(\mathbb{R}^3)$ is dense in $\mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$ w.r.t. $\|\cdot\|_{\mathrm{H}^1(\Omega)}$ and therefore also w.r.t. $\|\cdot\|_{\hat{\mathrm{H}}(\mathrm{curl},\Omega)}$.

Hence, it is left to show that every

$$f \in \nabla \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega) \cap \hat{\mathrm{H}}(\mathrm{curl}, \Omega)$$

can be approximated by a $\mathring{\mathrm{C}}^\infty_{\Gamma_0}(\mathbb{R}^3)$ function (w.r.t. $\|\cdot\|_{\hat{\mathrm{H}}(\mathrm{curl},\Omega)}$).

The following result is basically [Skr23a, Thm. 4.2].

Lemma 4.7. *Let $f \in \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$ such that $\nabla f \in \hat{\mathrm{H}}(\mathrm{curl}, \Omega)$ (in particular $\pi_\tau \nabla f \in \mathrm{L}^2_\tau(\partial\Omega)$). Then $\pi_\tau \nabla f = \nabla_\tau f\big|_{\partial\Omega}$ and $f \in W(\Omega) \cap \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$.*

Proof. Since $\nabla f \in \hat{\mathrm{H}}(\mathrm{curl}, \Omega)$, we know that $\pi_\tau \nabla f \in \mathrm{L}^2(\partial\Omega)$ which implies $f\big|_{\partial\Omega} \in \mathrm{H}^1(\partial\Omega)$ and $\nabla_\tau f\big|_{\partial\Omega} = \pi_\tau \nabla f$, see [Skr23a, Thm. 4.2]. Therefore, we conclude $f \in W(\Omega)$. ❑

This brings us to our first main theorem.

Theorem 4.8. *$\mathring{\mathrm{C}}^\infty_{\Gamma_0}(\mathbb{R}^3)$ is dense in $\hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$ w.r.t. $\|\cdot\|_{\hat{\mathrm{H}}(\mathrm{curl},\Omega)}$.*

Proof. Let $f \in \hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$ be arbitrary. Then we can decompose f into $f = f_1 + f_2$, where $f_1 \in \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$ and $f_2 \in \hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \nabla \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$.

By [BPS16, Lmm. 3.1] f_1 can be approximated by $\mathring{\mathrm{C}}^\infty_{\Gamma_0}(\mathbb{R}^3)$ functions w.r.t. $\|\cdot\|_{\mathrm{H}^1(\Omega)}$ and therefore also w.r.t. $\|\cdot\|_{\hat{\mathrm{H}}(\mathrm{curl},\Omega)}$.

By Lemma 4.7 we know that $f_2 \in W(\Omega) \cap \mathring{\mathrm{H}}^1_{\Gamma_0}(\Omega)$. Hence, we can apply Theorem 3.3 and obtain a sequence $(f_{2,n})_{n\in\mathbb{N}}$ that converges to f_2 w.r.t. $\|\cdot\|_{W(\Omega)}$. This gives

$$\begin{aligned}
&\|\nabla f_2 - \nabla f_{2,n}\|^2_{\hat{\mathrm{H}}(\mathrm{curl},\Omega)} \\
&\qquad = \|\nabla(f_2 - f_{2,n})\|^2_{\mathrm{L}^2(\Omega)} + \|\underbrace{\mathrm{curl}\,\nabla(f_2 - f_{2,n})}_{=0}\|^2_{\mathrm{L}^2(\Omega)} + \|\pi_\tau \nabla(f_2 - f_{2,n})\|^2_{\mathrm{L}^2(\partial\Omega)} \\
&\qquad \leq \|f_2 - f_{2,n}\|^2_{\mathrm{H}^1(\Omega)} + \big\|f_2\big|_{\partial\Omega} - f_{2,n}\big|_{\partial\Omega}\big\|^2_{\mathrm{H}^1(\partial\Omega)} \\
&\qquad = \|f_2 - f_{2,n}\|^2_{W(\Omega)} \to 0,
\end{aligned}$$

which finishes the proof. ❑

5. Density result without homogeneous part

Since we already know that $\mathring{\mathrm{C}}^\infty_{\Gamma_0}(\mathbb{R}^3)$ is dense in $\hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$, we can show the density of $\mathring{\mathrm{C}}^\infty(\mathbb{R}^3)$ in $\hat{\mathrm{H}}_{\Gamma_1}(\mathrm{curl}, \Omega)$ by a duality argument, which we will present in this section. This argument can be done in just a few lines within the notion of quasi Gelfand triples [Skr23b]. However, in order to stay relatively elementary we extract the essence and build a proof that avoids the introduction of this notion.

Basically we mimic the abstract boundary space for the tangential trace by $\mathring{\mathrm{H}}(\mathrm{curl}, \Omega)^\perp$, which can also be viewed as the boundary space as it is isometrically isomorphic.

Our standing assumption in this section is that (Ω, Γ_0) is Lipschitz pair and $\Gamma_1 \coloneqq \partial\Omega \setminus \overline{\Gamma_0}$.

Corollary 5.1. *If* $f \in \hat{\mathrm{H}}_{\Gamma_1}(\mathrm{curl}, \Omega)$*, then*

$$\langle \gamma_\tau f, \pi_\tau g \rangle_{\mathrm{L}^2(\Gamma_1)} = \langle \mathrm{curl}\, f, g \rangle_{\mathrm{L}^2(\Omega)} - \langle f, \mathrm{curl}\, g \rangle_{\mathrm{L}^2(\Omega)}$$

for all $g \in \hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$.

Proof. For $f \in \hat{\mathrm{H}}_{\Gamma_1}(\mathrm{curl}, \Omega)$ we have by definition

$$\langle \gamma_\tau f, \pi_\tau g \rangle_{\mathrm{L}^2(\Gamma_1)} = \langle \mathrm{curl}\, f, g \rangle_{\mathrm{L}^2(\Omega)} - \langle f, \mathrm{curl}\, g \rangle_{\mathrm{L}^2(\Omega)}$$

for all $g \in \mathring{\mathrm{C}}^\infty_{\Gamma_0}(\mathbb{R}^3)$. Since this equation is continuous in g w.r.t. $\|\cdot\|_{\hat{\mathrm{H}}(\mathrm{curl},\Omega)}$, we can extend it by continuity to $g \in \overline{\mathring{\mathrm{C}}^\infty_{\Gamma_0}(\mathbb{R}^3)}^{\hat{\mathrm{H}}(\mathrm{curl},\Omega)}$ and by Theorem 4.8 to $g \in \hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$. ❑

Lemma 5.2. *We have the following identity*

$$\mathring{\mathrm{H}}(\mathrm{curl}, \Omega)^\perp = \{ f \in \mathrm{H}(\mathrm{curl}, \Omega) \,|\, \mathrm{curl}\,\mathrm{curl}\, f = -f \},$$

where the orthogonal complement is taken in $\mathrm{H}(\mathrm{curl}, \Omega)$*, i.e., w.r.t.* $\langle \cdot, \cdot \rangle_{\mathrm{H}(\mathrm{curl},\Omega)}$*. Moreover,* curl *leaves the space* $\mathring{\mathrm{H}}(\mathrm{curl}, \Omega)^\perp$ *invariant.*

Proof. Note that by density of $\mathring{\mathrm{C}}^\infty(\Omega)$ in $\mathring{\mathrm{H}}(\mathrm{curl}, \Omega)$ both spaces have the same orthogonal complement. Hence,

$$\begin{aligned} f \in \mathring{\mathrm{H}}(\mathrm{curl}, \Omega)^\perp \quad &\Leftrightarrow \quad 0 = \langle f, g \rangle_{\mathrm{L}^2(\Omega)} + \langle \mathrm{curl}\, f, \mathrm{curl}\, g \rangle_{\mathrm{L}^2(\Omega)} \quad \forall\, g \in \mathring{\mathrm{C}}^\infty(\Omega) \\ &\Leftrightarrow \quad \mathrm{curl}\, f \in \mathrm{H}(\mathrm{curl}, \Omega) \quad \text{and} \quad \mathrm{curl}\,\mathrm{curl}\, f = -f. \end{aligned}$$

❑

Lemma 5.3. *Let* P *the orthogonal projection on* $\mathring{\mathrm{H}}(\mathrm{curl}, \Omega)^\perp$ *(in* $\mathrm{H}(\mathrm{curl}, \Omega)$*). Then* $\hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$ *is invariant under* P*, i.e.,* $f \in \hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$ *implies* $Pf \in \hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$.

Proof. Since $\mathrm{I} - P$ is the orthogonal projection on $\mathring{\mathrm{H}}(\mathrm{curl}, \Omega)$ and $\mathring{\mathrm{H}}(\mathrm{curl}, \Omega)$ is a subspace of $\hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$, we conclude that $(\mathrm{I} - P)f \in \hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$ for every $f \in \mathrm{H}(\mathrm{curl}, \Omega)$. Now for every $f \in \hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$ we have

$$Pf = f - (\mathrm{I} - P)f,$$

which is in $\hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$, since $\hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$ is a subspace. ❑

Lemma 5.4. *For every* $q \in \pi_\tau\big(\hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)\big)$ *there exists a* $g \in \mathring{\mathrm{H}}(\mathrm{curl}, \Omega)^\perp$ *such that*

$$\mathrm{curl}\, g \in \hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}(\mathrm{curl}, \Omega)^\perp \quad \textit{and} \quad \pi_\tau\, \mathrm{curl}\, g = q.$$

In particular,

$$\pi_\tau\big(\hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)\big) = \pi_\tau\big(\hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}(\mathrm{curl}, \Omega)^\perp\big).$$

Proof. By assumption we have $q = \pi_\tau f$ for $f \in \hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$. Let P denote the orthogonal projection on $\mathring{\mathrm{H}}(\mathrm{curl}, \Omega)^\perp$. Then by Lemma 5.3 we can decompose f into $f = Pf + (\mathrm{I} - P)f$, where both Pf and $(\mathrm{I} - P)f$ are also in $\hat{\mathrm{H}}(\mathrm{curl}, \Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl}, \Omega)$. Moreover, $(\mathrm{I} - P)f \in \mathring{\mathrm{H}}(\mathrm{curl}, \Omega)$, which gives $\pi_\tau(\mathrm{I} - P)f = 0$ and therefore

$$q = \pi_\tau f = \pi_\tau P f.$$

Since $Pf \in \mathring{\mathrm{H}}(\mathrm{curl}, \Omega)^\perp$, we have $\mathrm{curl}\,\mathrm{curl}\, Pf = -Pf$. Thus defining $g = -\,\mathrm{curl}\, Pf$ finishes the proof. ❑

Now we finally come to our second main theorem.

Theorem 5.5. $\mathring{\mathrm{C}}^\infty(\mathbb{R}^3)$ *is dense in* $\hat{\mathrm{H}}_{\Gamma_1}(\mathrm{curl}, \Omega)$ *w.r.t.* $\|\cdot\|_{\hat{\mathrm{H}}_{\Gamma_1}(\mathrm{curl},\Omega)}$.

Proof. By the definition of the norm of $\hat{\mathrm{H}}_{\Gamma_1}(\mathrm{curl},\Omega)$ the mapping $\gamma_\tau\colon \hat{\mathrm{H}}_{\Gamma_1}(\mathrm{curl},\Omega) \subseteq \mathrm{H}(\mathrm{curl},\Omega) \to \mathrm{L}^2_\tau(\Gamma_1)$ is closed. We define the following restriction of γ_τ

$$\hat{\gamma_\tau}\colon \begin{cases} \mathring{\mathrm{C}}^\infty(\mathbb{R}^3) \subseteq \mathrm{H}(\mathrm{curl},\Omega) & \to \quad \mathrm{L}^2_\tau(\Gamma_1), \\ f & \mapsto \quad \gamma_\tau f. \end{cases}$$

Since $\hat{\gamma_\tau} \subseteq \gamma_\tau$ we conclude

$$\hat{\gamma_\tau}^* \supseteq \gamma_\tau^*.$$

1. Step: *Calculate* $\operatorname{dom}\hat{\gamma_\tau}^*$. Let $q \in \operatorname{dom}\hat{\gamma_\tau}^*$. Then there exists a $g \in \mathrm{H}(\mathrm{curl},\Omega)$ such that

$$\langle \hat{\gamma_\tau} f, q\rangle_{\mathrm{L}^2(\Gamma_1)} = \langle f, g\rangle_{\mathrm{H}(\mathrm{curl},\Omega)} = \langle f, g\rangle_{\mathrm{L}^2(\Omega)} + \langle \mathrm{curl}\, f, \mathrm{curl}\, g\rangle_{\mathrm{L}^2(\Omega)} \tag{2}$$

for all $f \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^3)$. For $f \in \mathring{\mathrm{C}}^\infty_{\Gamma_1}(\mathbb{R}^3)$, we obtain

$$0 = \langle f, g\rangle_{\mathrm{L}^2(\Omega)} + \langle \mathrm{curl}\, f, \mathrm{curl}\, g\rangle_{\mathrm{L}^2(\Omega)},$$

which implies $\mathrm{curl}\, g \in \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl},\Omega)$ and $\mathrm{curl}\,\mathrm{curl}\, g = -g$, and by Lemma 5.2 $g \in \mathring{\mathrm{H}}(\mathrm{curl},\Omega)^\perp$. Hence, we revisit (2), where we extend q by 0 outside of Γ_1 in $\partial\Omega$

$$\langle \hat{\gamma_\tau} f, q\rangle_{\mathrm{L}^2(\partial\Omega)} = -\langle f, \mathrm{curl}\,\mathrm{curl}\, g\rangle_{\mathrm{L}^2(\Omega)} + \langle \mathrm{curl}\, f, \mathrm{curl}\, g\rangle_{\mathrm{L}^2(\Omega)}$$

for all $f \in \mathring{\mathrm{C}}^\infty(\mathbb{R}^3)$, which implies $\mathrm{curl}\, g \in \hat{\mathrm{H}}(\mathrm{curl},\Omega)$ and $q = \pi_\tau\,\mathrm{curl}\, g$. Consequently,

$$\begin{aligned}\operatorname{dom}\hat{\gamma_\tau}^* &\subseteq \pi_\tau\big(\hat{\mathrm{H}}(\mathrm{curl},\Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl},\Omega) \cap \mathring{\mathrm{H}}(\mathrm{curl},\Omega)^\perp\big) \\ &= \pi_\tau\big(\hat{\mathrm{H}}(\mathrm{curl},\Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl},\Omega)\big).\end{aligned}$$

2. Step: *Calculate* $\operatorname{dom}\gamma_\tau^*$. If $q \in \pi_\tau\big(\hat{\mathrm{H}}(\mathrm{curl},\Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl},\Omega)\big)$, then by Lemma 5.4 there exists a $g \in \mathring{\mathrm{H}}(\mathrm{curl},\Omega)^\perp$ such that $\mathrm{curl}\, g \in \hat{\mathrm{H}}(\mathrm{curl},\Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl},\Omega)$ and $\pi_\tau\,\mathrm{curl}\, g = q$. Hence, by Corollary 5.1 for $f \in \hat{\mathrm{H}}_{\Gamma_1}(\mathrm{curl},\Omega)$ and $\mathrm{curl}\, g$ we have

$$\langle \gamma_\tau f, \underbrace{\gamma_\tau\,\mathrm{curl}\, g}_{=q}\rangle_{\mathrm{L}^2(\Gamma_1)} = \langle \mathrm{curl}\, f, \mathrm{curl}\, g\rangle_{\mathrm{L}^2(\Omega)} - \langle f, \underbrace{\mathrm{curl}\,\mathrm{curl}\, g}_{=-g}\rangle_{\mathrm{L}^2(\Omega)} = \langle f, g\rangle_{\mathrm{H}(\mathrm{curl},\Omega)},$$

which implies $q \in \operatorname{dom}\gamma_\tau^*$. Consequently,

$$\operatorname{dom}\gamma_\tau^* \supseteq \pi_\tau\big(\hat{\mathrm{H}}(\mathrm{curl},\Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl},\Omega)\big).$$

3. Step: Combining the results of the previous steps yields

$$\begin{aligned}\pi_\tau\big(\hat{\mathrm{H}}(\mathrm{curl},\Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl},\Omega)\big) &\supseteq \operatorname{dom}\hat{\gamma_\tau}^* \\ &\supseteq \operatorname{dom}\gamma_\tau^* \supseteq \pi_\tau\big(\hat{\mathrm{H}}(\mathrm{curl},\Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl},\Omega)\big).\end{aligned}$$

Hence, $\hat{\gamma_\tau}^* = \gamma_\tau^*$ and therefore

$$\overline{\hat{\gamma_\tau}} = \hat{\gamma_\tau}^{**} = \gamma_\tau^{**} = \gamma_\tau,$$

which implies $\mathring{\mathrm{C}}^\infty(\mathbb{R}^3)$ is dense in $\hat{\mathrm{H}}_{\Gamma_1}(\mathrm{curl},\Omega)$ w.r.t. the graph norm of γ_τ with is $\|\cdot\|_{\hat{\mathrm{H}}_{\Gamma_1}(\mathrm{curl},\Omega)}$. ❑

6. Conclusion

We have defined $\mathrm{H}(\mathrm{curl},\Omega)$ fields that possess an L^2 tangential trace on $\Gamma_1 \subseteq \partial\Omega$ via a weak approach (by representation in the $\mathrm{L}^2(\Gamma_1)$ inner product) and showed that the C^∞ fields are dense in this space. This is a generalization of [BBBCD97], where the case $\Gamma_1 = \partial\Omega$ was regarded. In fact for partial tangential traces there is the second question about the density with additional homogeneous boundary conditions on $\Gamma_0 = \partial\Omega \setminus \overline{\Gamma_1}$. This was exactly the open problem in [WS13, Sec. 5], which we could solve. In particular they were asking whether $\mathrm{H}^1_{\Gamma_0}(\Omega)$ is dense in $\hat{\mathrm{H}}(\mathrm{curl},\Omega) \cap \mathring{\mathrm{H}}_{\Gamma_0}(\mathrm{curl},\Omega)$, which is in fact a weaker version of Theorem 4.8.

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TU Bergakademie Freiberg, Institute of Applied Analysis, Akademiestrasse 6, D-09596 Freiberg, Germany
Email address: nathanael.skrepek@math.tu-freiberg.de

Technische Universität Dresden (TUDD), Fakultät Mathematik, Institut für Analysis, Zellescher Weg 12-14, 01069 Dresden, Germany
Email address: dirk.pauly@tu-dresden.de