19.04.2024

Exercise Sheet 1

Exercise 1. Let $k, m > 0$. Solve the following system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{m} & 0 \\ -k & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad t \geq 0,$$
$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$ ✧

Exercise 2. Let X be the vector space of all complex functions on the unit interval $[0,1]$. For any $x \in X$ consider the function $p_x \colon X \to \mathbb{R}$ defined by

$$p_x(f) = |f(x)|, \quad f \in X.$$

(a) Prove that $\mathcal{P} := \{p_x \,|\, x \in [0,1]\}$ is a separating family of seminorms on X.

(b) Show that the locally convex topology on X induced by the family of seminorms $\mathcal{P}$ is the topology of pointwise convergence on X. ✧

Exercise 3. Let $\Omega = (-1,1)$. Compute the distributional derivative of the following functions

(a) $f(x) = |x|$, for $x \in \Omega$,

(b) $f(x) = x^3 + x + 1$, for $x \in \Omega$,

(c) $f(x) = \mathbb{1}_{(0,1)}(x)$, for $x \in \Omega$. ✧

Exercise 4. Let Ω be an open subset of $\mathbb{R}$. Let $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ and $\psi \in \mathrm{C}^\infty(\Omega)$. Prove that

$$(\psi f)' = \psi f' + \psi' f \quad \text{(in the distributional sense).}$$ ✧

Exercise Sheet 2

Exercise 1. Let $\Omega \subseteq \mathbb{R}^d$ be open, $\Lambda \in \mathcal{D}'(\Omega)$ and $\psi \in \mathrm{C}^\infty(\Omega)$. Then show that

(a) $\mathrm{D}^\alpha \mathrm{D}^\beta \Lambda = \mathrm{D}^\beta \mathrm{D}^\alpha \Lambda$ for all $\alpha, \beta \in \mathbb{N}_0^d$,

(b) $\mathrm{D}^\alpha \mathrm{D}^\beta \Lambda = \mathrm{D}^{\alpha+\beta} \Lambda$ for all $\alpha, \beta \in \mathbb{N}_0^d$,

(c) $\mathrm{D}^\alpha(\psi\Lambda) = \psi \mathrm{D}^\alpha \Lambda + (\mathrm{D}^\alpha \psi)\Lambda$ for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| = 1$. ✧

Exercise 2. Let $\Omega \subseteq \mathbb{R}^d$ be open and endow $\mathcal{D}(\Omega)$ with topology that is induces by the family of seminorms $P := \{\|\cdot\|_n \mid n \in \mathbb{N}\}$, where

$$\|\phi\|_n = \sup_{|\alpha| \leq n} \sup_{x \in \Omega} |\mathrm{D}^\alpha \phi(x)|.$$

Show that the topology that is induced by P is metrizable, i.e., there exists a metric that induces the same topology. What is the difference between this new topology and the topology that we have introduced in the lecture for $\mathcal{D}(\Omega)$?

Hint: Think of a sequence that converges to 0 w.r.t. to the new topology that does not converge to 0 w.r.t. the topology of the lecture. Why are these additional convergent sequences undesired? ✧

Exercise 3. Let $\Omega \subseteq \mathbb{R}^d$ be open. Find a distribution $\Lambda \in \mathcal{D}'(\Omega)$ that is not the derivative of a regular distribution, namely such that there does not exist any function $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ and any $\alpha \in \mathbb{N}_0^d$ such that $\Lambda = \mathrm{D}^\alpha \Lambda_f$.

Hint: For $\Omega = \mathbb{R}$, construct a distribution using Dirac distributions and their derivatives. ✧

Exercise 4. Prove the following lemmata.

(a) Let $(f_m)_{m\in\mathbb{N}}$ be a sequence in $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$ that converges pointwise to $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ such that for every compact $K \subseteq \Omega$ there exists an integrable function g_K such that $|f_m(x)| \leq g_K(x)$ for a.e. $x \in K$. Then f_m converges to f in $\mathcal{D}'(\Omega)$, i.e.,

$$\lim_{m\in\mathbb{N}} \langle f_m, \phi\rangle = \langle f, \phi\rangle$$

for all $\phi \in \mathcal{D}(\Omega)$.

(b) Let $(\Lambda_m)_{m\in\mathbb{N}}$ be a sequence in $\mathcal{D}'(\Omega)$ that converges to $\Lambda \in \mathcal{D}'(\Omega)$ in $\mathcal{D}'(\Omega)$. Then $\mathrm{D}^\alpha \Lambda_m$ converges to $\mathrm{D}^\alpha \Lambda$ in $\mathcal{D}'(\Omega)$ for every $\alpha \in \mathbb{N}_0^d$. ✧

Exercise 5. Let $\hat{\rho}\colon \mathbb{R}^d \to \mathbb{R}$ be the function defined by

$$\hat{\rho}(x) = \begin{cases} \exp\left(\frac{1}{\|x\|_2^2 - 1}\right), & \|x\|_2 < 1, \\ 0, & \text{else.} \end{cases}$$

Moreover, consider the functions $\rho := \frac{1}{\|\hat{\rho}\|_{\mathrm{L}^1}} \hat{\rho}$ and

$$\rho_\epsilon(x) := \epsilon^{-d} \rho\left(\frac{x}{\epsilon}\right).$$

Show that ρ_ϵ converges to the Dirac distribution δ_0 in $\mathcal{D}'(\Omega)$ as $\epsilon \to 0$ for any open $\Omega \subseteq \mathbb{R}^d$ that contains 0. ✧

31.05.2024

Exercise Sheet 3

Exercise 1. Prove that every classical solution is also a distributional solution. ✧

Exercise 2. Prove that every regular distribution can be approximated with test functions with respect to the weak*-topology.
Hint: Use cut-off functions and mollifiers as in the proof of Corollary 2.1.5. ✧

Exercise 3. Prove that the space of test functions is dense in the space of distributions with respect to the weak*-topology.
Hint: Use first that every distribution is locally the derivative of a regular distribution. Then, combine this with Exercise 2. ✧

Exercise 4. Check that the function

$$u_0(x) = -\frac{1}{2\pi} \ln |x| = -\frac{1}{2\pi} \ln\Big(\sqrt{x_1^2 + x_2^2}\Big)$$

is the fundamental solution of the differential operator $L = \Delta$ in $\mathbb{R}^2$.
Hint: First show that $Lu_0(x) = 0$ for $x \neq 0$. Then show that for $\Omega_\epsilon := \Omega \setminus \overline{\mathrm{B}}_\epsilon(0)$ that

$$\lim_{\epsilon \to 0} \int_{\Omega_\epsilon} u_0 L\phi \, \mathrm{d}\lambda = \phi(0) \quad \forall \phi \in \mathcal{D}(\mathbb{R}^2).$$

This can be done by the integration by parts formula in $\mathbb{R}^2$. Note that u_0 is constant on $\partial\Omega_\epsilon$ for every fixed ϵ. ✧

Exercise 5. Show that

$$u_0(x) = \frac{|x|^2}{8\pi} \ln(|x|) = \frac{x_1^2 + x_2^2}{8\pi} \ln\Big(\sqrt{x_1^2 + x_2^2}\Big)$$

is a fundamental solution for $L = \Delta^2$ in $\mathbb{R}^2$. ✧

Exercise Sheet 4

Exercise 1. Show that the following sets are Lipschitz domains

(a) $[0,1] \times [0,1]$

(b) $\mathrm{B}_1(0) \subseteq \mathbb{R}^2$ ✧

Exercise 2. Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz domain, $f \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d)^d$ and $g \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d)$. Show the integration by parts formula

$$\int_\Omega (\operatorname{div} f) g \,\mathrm{d}\lambda + \int_\Omega f \cdot \nabla g \,\mathrm{d}\lambda = \int_{\partial\Omega} \nu \cdot f g \,\mathrm{d}\mu\,.$$

Moreover, show that

$$\operatorname{div} f(x) = \lim_{r\to 0} \frac{1}{\lambda(\mathrm{B}_r(x))} \int_{\partial \mathrm{B}_r(x)} \nu \cdot f \,\mathrm{d}\mu$$

What does this mean? In context of sink or source?
Hint: Find a product rule for div. ✧

The curl operator for a vector field $E \in \mathrm{C}^1(\mathbb{R}^3)^3$ is defined by

$$\operatorname{curl} E = \nabla \times E = \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} -\partial_3 E_2 + \partial_2 E_3 \\ \partial_3 E_1 - \partial_1 E_3 \\ -\partial_2 E_1 + \partial_1 E_2 \end{bmatrix}$$

Exercise 3. Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain, $E, H \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^3)$. Show that

$$\begin{aligned} \int_\Omega \operatorname{curl} E \cdot H \,\mathrm{d}\lambda - \int_\Omega E \cdot \operatorname{curl} H \,\mathrm{d}\lambda &= \int_{\partial\Omega} \nu \times E \cdot H \,\mathrm{d}\mu \\ &= \int_{\partial\Omega} \big[\nu \times E\big] \cdot \big[(\nu \times H) \times \nu\big] \,\mathrm{d}\mu\,. \end{aligned}$$

Moreover, show that

$$\operatorname{curl} E(x) = \lim_{r\to 0} \frac{1}{\lambda(\mathrm{B}_r(x))} \int_{\partial \mathrm{B}_r(x)} \nu \times E \,\mathrm{d}\mu$$

What is the geometrical meaning of $\nu \times E$, $(\nu \times H) \times \nu$ and limit representation of $\operatorname{curl} E(x)$?
Hint: Look at the integrals component-wise in order to apply Stokes' theorem. ✧

We define

$$\mathrm{C}^\infty(\overline{\Omega}) := \big\{ f\big|_\Omega \,\big|\, f \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^d) \big\}$$

Recall that

$$\mathrm{B}_\epsilon(\Omega) = \bigcup_{x\in\Omega} \mathrm{B}_\epsilon(x).$$

Exercise 4. Let Ω be bounded. Show for $\epsilon > 0$ that

$$\mathrm{C}^\infty(\overline{\Omega}) = \big\{ f\big|_\Omega \,\big|\, f \in \mathrm{C}^\infty(\mathbb{R}^d) \big\} = \big\{ f\big|_\Omega \,\big|\, f \in \mathrm{C}^\infty\big(\mathrm{B}_\epsilon(\Omega)\big) \big\}.$$ ✧

Exercise 5. Let X, Y be Banach spaces and $A\colon X \to Y$ a bounded linear mapping, i.e., $A \in \mathcal{L}_\mathrm{b}(X,Y)$. Show that: if $\|Ax\| \geq c\|x\|$ for a $c > 0$, then A is injective and $\operatorname{ran} A$ is closed. ✧

Exercise Sheet 5

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain, A be a measurable matrix-valued function such that $c^{-1} \leq A \leq c$ for a $c > 0$ and $L := -\operatorname{div} A\nabla$.

Exercise 1. Let $b\colon \mathring{\mathrm{H}}^1(\Omega) \times \mathring{\mathrm{H}}^1(\Omega) \to \mathbb{C}$ be the sesquilinear form defined by

$$b(u,v) = \langle A\nabla u, \nabla v\rangle_{L^2(\Omega)} \quad \forall u, v \in \mathring{\mathrm{H}}^1(\Omega).$$

Prove that b is coercive, i.e., it exists a $C > 0$ such that $b(u,u) \geq C\|u\|^2_{\mathring{\mathrm{H}}^1(\Omega)}$ for all $u \in \mathring{\mathrm{H}}^1(\Omega)$. ✧

Exercise 2. Consider a weak solution u of $Lu = f$ in Ω and $u = 0$ on $\partial\Omega$. Prove that u is a distributional solution. ✧

Exercise 3. Find the weak formulation of $Lu = f$ in Ω and $\nu \cdot A\nabla u = 0$ (Neumann boundary conditions). ✧

Exercise 4. Show that the following maps are continuous:

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\colon \mathrm{C}^1(\mathbb{R}) \subseteq \mathrm{L}^2(\mathbb{R}) \to \mathrm{H}^{-1}(\mathbb{R})$;

(b) $\operatorname{div}\colon \mathrm{C}^1(\Omega) \subseteq \mathrm{L}^2(\Omega) \to \mathrm{H}^{-1}(\Omega)$. ✧

Exercise 5. (a) Show that $\mathrm{H}^{-1}(\Omega)$ is a Hilbert space;

(b) Consider the map $\Phi\colon \mathrm{H}^{-1}(\Omega) \to \mathring{\mathrm{H}}^1(\Omega)$ such that

$$\langle \Phi f, g\rangle_{\mathring{\mathrm{H}}^1(\Omega)} = \langle f, g\rangle_{\mathrm{H}^{-1}(\Omega),\mathring{\mathrm{H}}^1(\Omega)} \quad \forall f \in \mathrm{H}^{-1}(\Omega)\, \forall g \in \mathring{\mathrm{H}}^1(\Omega).$$

Prove that the Riesz isomorphism Φ is unitary, i.e., $\Phi^* = \Phi^{-1}$. ✧

12.07.2024

Exercise Sheet 6

Exercise 1. We regard the one-dimension Laplacian equation on the interval $(0,1)$, which is the following differential equation

$$\begin{aligned} -u''(x) &= 1, \\ u(0) = u(1) &= 0, \end{aligned} \qquad \text{or weakly} \qquad \langle u', \phi' \rangle_{\mathrm{L}^2(0,1)} = \langle 1, \phi \rangle_{\mathrm{L}^2(0,1)} \quad \forall \phi \in \mathring{\mathrm{H}}^1(0,1).$$

Furthermore, we regard the finite dimensional subspace $V_{\frac{1}{4}}$ that is spanned by the functions u_1, u_2 and u_3, which are given by

$$u_j \colon \begin{cases} (0,1) \to \mathbb{C}, \\ x \mapsto \begin{cases} 0, & x \le t_{j-1}, \\ 4(x - t_{j-1}), & t_{j-1} < x \le t_j, \\ 1 - 4(x - t_j), & t_j < x \le t_{j+1}, \\ 0, & t_{j+1} < x, \end{cases} \end{cases}$$

where $t_1 = 1/4$, $t_2 = 1/2$ and $t_3 = 3/4$. In Figure 1 we see the graphs of the functions u_1, u_2 and u_3.

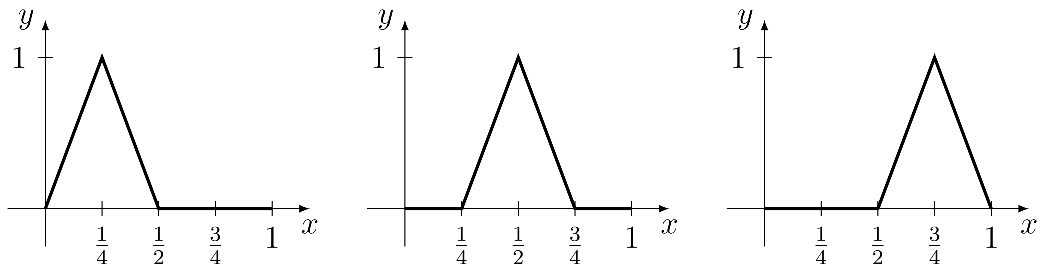

Figure 1: Illustration of the functions u_1, u_2 and u_3.

(a) Use the Galerkin method to find an approximation of the solution.

(b) Calculate the actual solution of the problem and compare it to the approximation. ✧

Exercise 2. Show that $\frac{\mathrm{d}}{\mathrm{d}x} \colon \mathrm{H}^1(0,1) \subseteq \mathrm{L}^2(0,1) \to \mathrm{L}^2(0,1)$ is a densely defined and closed operator. Moreover,

(a) show $\frac{\mathrm{d}}{\mathrm{d}x}$ is not bounded,

(b) and calculate the adjoint of $\frac{\mathrm{d}}{\mathrm{d}x}$. ✧

Exercise 3. On the space $X = \mathrm{L}^2(\mathbb{R})$ we consider the so-called *shift semigroup* $T \colon [0, +\infty) \to \mathcal{L}_{\mathrm{b}}(X)$ defined by

$$T(t)f(s) = f(s+t), \quad \forall s \in \mathbb{R},\ t \in [0, +\infty),\ f \in X.$$

Show that

(a) T is strongly continuous semigroup.

(b) T is a contraction semigroup.

(c) the infinitesimal generator of T is given by $Af := f'$ with domain

$$\operatorname{dom}(A) = \mathrm{H}^1(\mathbb{R}).$$ ✧

Exercise 4. Let $A\colon \operatorname{dom} A \subseteq X \to X$ be densely defined and closed operator on a Hilbert space X that is diagonalizable, i.e., there exists a orthonormal basis $(\phi_n)_{n\in\mathbb{N}}$ of X and a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ such that

$$A\phi_n = \alpha_n \phi_n.$$

Show that A generates a strongly continuous semigroup, if there exists a $C \in \mathbb{R}$ such that $\operatorname{Re} \alpha_i \leq C$.
Hint: Think of the action of the semigroup on the eigenvectors ϕ_i and conclude then by linearity the action on an arbitrary vector $x \in X$. ✧

Exercise 5. Prove that the Laplace operator

$$\Delta\colon \mathrm{H}^2(\Omega) \cap \mathring{\mathrm{H}}^1(\Omega) \subseteq \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$$

is maximally dissipative and conclude that the heat equation

$$\begin{aligned} \partial_t u(t,x) &= \Delta u(t,x), && x \in \Omega, \quad t \geq 0, \\ u(t,x) &= 0, && x \in \partial\Omega,\ t \geq 0, \\ u(0,x) &= u_0(x), && x \in \Omega \end{aligned}$$

has a unique solution that satifies $\|u(t,\cdot)\|_{\mathrm{L}^2(\Omega)} \leq \|u_0\|_{\mathrm{L}^2(\Omega)}$ for all $t \geq 0$.
Hint: In order to show maximality show that Δ (on the given domain) is self-adjoint ✧